STATE VARIABLE APPROACHES

TO

LINEAR ELECTRICAL NETWORK ANALYSIS

BY

TANZO NITTA

SEPTEMBER 1977

FACULTY OF ENGINEERING

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CONTENTS

CHAPTER 1　INTRODUCTION

In analizing electro-magnetic phenomena, there are three steps to carry out:

First step: Finding a equivalent network for a electric apparatus.

Second step: Finding equations describing the network.

Third step: Finding the solutions of the equations.

This paper is mainly concerned with the second step. The first and the third steps are considered to a small extent.

For networks which contain linear passive resistors, inductors and capacitors (RCL networks), Emeritus Proffesor S.Hayashi showed a method of formulating a set of equations, which be called the fundamental equations. The variables in the fundamental equations are the voltages across the capacitors and the currents through the inductors. The fundamental equations are simultaneous differential equations of the first order. However, they are not always independent.

In 1950's, the description of a network by means of the state equations,

$\dot{x}=Ax+Bu$,

is proposed for RCL networks by T.R.Bashkow. After that, a great many studies on systematic formulations of the state equations for various networks are published. It is called state variable approach to network analysis. The state variables are closely relative to the network topology. The state variable approach is convenient for computer-aided network analysis.

In Chapter 2 - 5, the state variable approach to various networks is studied. In Chapter 2, the state variable approach to RLC networks is described for later use.

In Chapter 3, is studied systematic formulations of state equations for linear, reciprocal, passive, time-invariant networks with mutual coupling elements. These networks are equivalent to those which contain resistors, inductors, capacitors, ideal-transformers and independent sources (RCLT networks). The properties of the state equations are also studied.

In Chapter 4, are studied the solvability and a systematic formulation of the state equations for linear, nonreciprocal passive time-invariant networks. These networks are equivalent to those which contain resistors, capacitors, gyrators and independent sources (RCG networks).

In Chapter 5, the solvability and a systematic formulation of state equations for linear, active, time-invariant networks (active networks) is presented.

In Chapter 6, networks containing periodically operated switches are considered by use of the results of Chapter 2 - 5. The networks have important applications. These networks are studied from a point of view different from that by S.Hayashi.

CHAPTER 2 STATE VARIABLE APPROACH

2-1 INTRODUCTION

In this chapter, graph theory and its terminology necessary to the state variable approach of lumped networks are described. A formulation of explicit state equations of linear passive lumped networks cotaining resistors, inductors, capacitors and independent sources as network-elements (so called RLC networks) is described, as a preparation for formulating explicit state equations of more general linear lumped networks.

2-2 NETWORK AND GRAPH

Consider a lumped network, then the graph of the network is obtained by replacing each lumped two terminal element with an oriented branch. The graph obtained from a graph G by contracting a branch b"a set of branches S"[@] is denoted by $G[b]$"$G[S]$". The graph obtained from G by deleting a branch b"a set of branches S" is denoted by $G\{b\}$"$G\{S\}$". The graph obtained from G by contracting a branch b_1"a set of branches S_1" and deleting a branch b_2"a set of branches S_2" is denoted by $G\{b_1;b_2\}$"$G[S_1;S_2\}$".

A branch of a graph corresponding to a capacitor"resistor, inductor, voltage source, current source" is called a capacitor-" resistor-, inductor- voltage source-, current source-" branch.

A network and its graph are, respectively, denoted by N^* and G^*. The graph[voltage souce-branches;current source-branches} is denoted by G, and its corresponding network is denoted by N.

@ The repetition of the (dual) sentence is avoided. The word in " " are to replace their preceding words in the (dual) sentence.

The rank"nullity" of G is denoted by $r(G)$"$\mu(G)$".

A tree and its corresponding cotree are denoted by T and $\bar{T}$, respectively.

A branch of T"$\bar{T}$" is called a twig"a link".

A fundamental loop"cut-set" uniquely determined by a link l and some twigs"a twig t and some links" is denoted by *loop(l)*"*cut-set(t)*"

An *(i,j)*element of a matrix A is denoted by *a(i,j)*. The cofactor of *a(i,j)* is denoted by *Cof.A(i,j)*. The determinant of A is denoted by |A|.

Let B and Q denote basic loop and basic cut-set matrices, respectively, then nonsingular majar submatrices of B"Q" are in one to one correspondence with cotrees"trees".

A graph is assumed to posses n_n nodes and n_b branches. A fundamental loop matrix of G for a tree T is written as

$$B=[1,F], \tag{2-2.1}$$

where the matrix is a $\mu(G) \times n_b$ matrix, 1 indicates the unit matrix of the order $\mu(G)$ and F is known to be a unimodular matrix. Similarly, a fundamental cut-set matrix of G for T is written as

$$Q=[-F',1], \tag{2-2.2}$$

where the matrix Q is an $r(G) \times n_b$ matrix and 1 indicates the unit matrix of the order $r(G)$.

The matrix F is called the characteristic part of a fundamental matrix for a tree T. Every row"column" of F corresponds to a link "a twig".

The i-th row"the j-th column" of a matrix A is denoted by a(i) "a$\{j\}$".

If by exchanging a link l_1 with a twig t_1 for T, another tree,

$T-t_1+l_1$, can be obtained, such a branch-exchange is called the elementary tree-transformation. A tree, $T_1=T-(t_1+\cdots+t_m)+(l_1+\cdots+l_m)$, is a tree obtained by m elementary tree-transformations from a tree T.

Lemma 2-2.1

Assume that the fundamental loop matrix B"the fundamaental cut-set matrix Q" for a tree T is given by B=[1,F]"Q=[-F',1]". If a tree T_1 is obtained by m tree-transformations from T such as $T_1=T-(t_1+\cdots+t_m)+(l_1+\cdots+l_m)$, and the matrix $[B_1,B_2]$"$[Q_1,Q_2]$" is obtained by permutating columns corresponding to t_i and l_i $\{i=1,\cdots,m\}$ of B"Q", then next equality holds,

$$|Q_2|\cdot|B_1|=(-1)^m. \tag{2-2.3}$$

Proof: If the matrix B is written as

$$B=\begin{vmatrix} 1, & & , a_{11}, \cdots, a_{1r} \\ \cdot & 0 & \cdot \quad\quad \cdot \\ & \cdot & \cdot \quad\quad \cdot \\ 0 & \cdot & \cdot \quad\quad \cdot \\ & & 1, a_{\mu 1}, \cdots, a_{\mu r} \end{vmatrix}, \tag{2-2.4}$$

then Q is written as

$$Q=\begin{vmatrix} -a_{11}, \cdots, -a_{\mu 1}, 1, & & \\ \cdot \quad\quad \cdot & \cdot & 0 \\ \cdot \quad\quad \cdot & & \cdot \\ \cdot \quad\quad \cdot & 0 & \cdot \\ -a_{1r}, \quad , -a_{\mu r}, & & 1 \end{vmatrix}, \tag{2-2.5}$$

where $a_{ij}=\pm 1, 0$.

Then the matrices B_1 and Q_2 may be written as

$$B_1=\begin{vmatrix}
1, & , a_{1j_1}, & \cdots, & a_{1j_2}, & \cdots\cdots, & a_{1j_m} & & \\
\cdot & \cdot & & \cdot & & \cdot & , & \\
& \cdot\ \cdot & & \cdot & & \cdot & & \\
& , a_{i_1j_1}, & \cdots, & a_{i_1j_2}, & \cdots\cdots, & a_{i_1j_m}, & & \\
& \cdot & 1 & \cdot & & \cdot & & \\
& \cdot & \cdot & \cdot & & \cdot & & \\
0 & , a_{i_2j_1}, & \cdots, & a_{i_2j_2}, & \cdots\cdots, & a_{i_2j_m}, & 0 & \\
& \cdot & & \cdot & 1, & \cdot & & \\
& \cdot & & \cdot & \cdot & \cdot & & \\
& , a_{i_mj_1}, & \cdots, & a_{i_mj_2}, & \cdots\cdots, & a_{i_mj_m}, & & \\
& \cdot & & \cdot & & \cdot & \cdot & \\
& , a_{\mu j_1}, & \cdots, & a_{\mu j_2}, & \cdots\cdots, & a_{\mu j_m}, & & 1
\end{vmatrix} \qquad (2\text{-}2.6)$$

$$Q_2=\begin{vmatrix}
1, & , -a_{i_1 1}, & \cdots, & -a_{i_2 1}, & \cdots\cdots, & -a_{i_m 1}, & & \\
\cdot & \cdot & & \cdot & & \cdot & & \\
& \cdot\ \cdot & & \cdot & & \cdot & & \\
& , -a_{i_1j_1}, & & , -a_{i_2j_1}, & & , -a_{i_mj_1}, & & \\
& \cdot & 1 & \cdot & & \cdot & & \\
0 & \cdot & \cdot & \cdot & & \cdot & 0 & \\
& , -a_{i_1j_2}, & \cdots, & -a_{i_2j_2}, & \cdots\cdots, & -a_{i_mj_2}, & & \\
& \cdot & & \cdot & 1 & \cdot & & \\
& \cdot & & \cdot & \cdot & \cdot & & \\
& , -a_{i_1j_m}, & \cdots, & -a_{i_2j_m}, & \cdots\cdots, & -a_{i_mj_m}, & & \\
& \cdot & & \cdot & & \cdot & \cdot & \\
& , -a_{i_1r}, & \cdots, & -a_{i_2r}, & \cdots\cdots, & -a_{i_mr}, & & 1
\end{vmatrix} . \qquad (2\text{-}2.7)$$

Then the following equality holds

$$|Q_2||B_1|=\begin{vmatrix}
-a_{i_1j_1}, & -a_{i_2j_2}, & \cdots\cdots, & -a_{i_mj_1} \\
\cdot & \cdot & \cdots\cdots & \cdot \\
-a_{i_1j_m}, & -a_{i_2j_m}, & \cdots\cdots, & -a_{i_mj_m}
\end{vmatrix}
\begin{vmatrix}
a_{i_1j_1}, & a_{i_1j_2}, & \cdots, & a_{i_1j_m} \\
\cdot & \cdot & \cdots & \cdot \\
a_{i_mj_1}, & a_{i_mj_2}, & \cdots, & a_{i_mj_m}
\end{vmatrix}$$

$$=(-1)^m \begin{vmatrix} a_{i_1 j_1}, a_{i_1 j_2}, \cdots, a_{i_1 j_m} \\ \cdot \quad \cdot \quad \cdots \quad \cdot \\ a_{i_m j_1}, a_{i_m j_2}, \cdots, a_{i_m j_m} \end{vmatrix}^2$$

$$=(-1)^m. \tag{2-2.8}$$

Q.E.D

The number of elements which are contained in a set S is denoted by $|S|$.

The set of twig-"link-"capacitor-branches, twig-"link-"resistor-branches and twig-"link-"inductor-branches are, respectively, denoted by $T(C)$"$\bar{T}(C)$", $T(R)$"$\bar{T}(R)$" and $T(L)$"$\bar{T}(L)$".

Definition 2-2.1

A normal tree (as denoted by T_N) for an RLC network is a tree of G such that $|T(C)|+|\bar{T}(L)|$ is a maximum.

2-3 STATE EQUATION

Assumption A

In G^*, there exists neither loop composed of only independent voltage source-branches nor cut-set composed of only independent current source-branches.

In a network which does not satisfy Assumption A, the Kirchhoff's voltage"current" law may not holds. In the Kirchhoff's voltage and current laws hold in this network, the voltages"currents" of independent voltage"current" sources in such a loop"cut-set" are linearly dependent. The network obtained by deleting"contracting" any one independent voltage"current" source in the loop"cut-set" is equivalent to the original network. Its graph satisfies Assumption A. Then consider an RLC network which satisfies Assumption A.

Let v and i denote the branch voltage and the branch current respectively. The i-th components of v and i are voltage across the i-th branch and current through the i-th branch. Let a fundamental loop and a fundamental cut-set matrices for a tree T be B and Q, then the Kirchhoff's voltage and current laws state

$$Bv=e \qquad (2\text{-}3.1)$$

$$Qi=j, \qquad (2\text{-}3.2)$$

where e is a loop source voltage vector, the i-th component of which is the algebraic sum of source voltages appearing in the i-th fundamental loop, and j is a cut-set source current vector, the k-th component of which is the algebraic sum of source currents appearing in the k-th fundamental cut-set.

The relation between branch voltages and branch currents may be represented by

$$Yv=Zi, \qquad (2\text{-}3.3)$$

where Y and Z are diagonal matrices. The each (k,k) component of Y and Z is specified as follows.

$y(k,k)=1,\quad z(k,k)=r_k$ for a resistor-branch

$y(k,k)=C_k p,\quad z(k,k)=1$ for a capacitor-branch

$y(k,k)=1,\quad z(k,k)=L_k p$ for an inductor-branch,

where $p\equiv d/dt$.

Eq.(2-3.1),(2-3.2) and (2-3.3) may be written as a matric equation:

$$\begin{vmatrix} B & 0 \\ 0 & Q \\ Y & -Z \end{vmatrix} \begin{vmatrix} v \\ i \end{vmatrix} = \begin{vmatrix} e \\ j \\ 0 \end{vmatrix}, \qquad (2\text{-}3.4)$$

which is called a network equation. The coefficient matrix is denoted by $\theta(p)$.

It is known that if and only if the matrix $\theta(p)$ is nonsingular, Eq.(2-3.4) has a unique solution. The unique solvability of network means that Eq.(2-3.4) has a unique solution. Whether the network has a unique solution or not is the problem on network solvability.

The next theorem has already obtained.

Theorem 2-3.1

An RLC network which satisfies Assumption A has a unique solution.

Definition 2-3.1[1]

The natural angular frequencies of a network are defined to be the roots of the characteristic polynominal of $\theta(p)$. The number of these natural angular frequencies (denoted by σ) is called the order of complexity of the network.

Theorem 2-3.2[1]

The order of complexity σ of an RLC network is given by

$$\sigma=|T_N(C)|+|\bar{T}_N(L)|. \qquad (2\text{-}3.5)$$

A method of formulating the state equations of RLC networks been already given[2][3]. Let us descrive it briefly for the studies in latter chapters.

The branch voltages and currents may be, conveniently, partitioned as

$$v = \begin{vmatrix} v_S \\ v_R \\ v_L \\ v_C \\ v_G \\ v_\Gamma \end{vmatrix} \qquad i = \begin{vmatrix} i_S \\ i_R \\ i_L \\ i_C \\ i_G \\ i_\Gamma \end{vmatrix}, \tag{2-3.6}$$

where the subscripts S, R and L denote link-capacitors, link-resistors and link-inductors, respectively. The subscript C, G and Γ denote twig-capacitors, twig-resistors and twig-inductors, respectively.

The characteristic part of a fundamental loop matrix for T_N may be represented by

$$\begin{vmatrix} F_{SC}, & 0, & 0 \\ F_{RC}, & F_{RG}, & 0 \\ F_{LC}, & F_{LG}, & F_{L\Gamma} \end{vmatrix} \tag{2-3.7}$$

The relations between branch voltages and currents can be expressed as

$$\begin{vmatrix} C_1, & 0 \\ 0, & C_2 \end{vmatrix} \frac{d}{dt} \begin{vmatrix} v_S \\ v_C \end{vmatrix} = \begin{vmatrix} i_S \\ i_C \end{vmatrix}, \quad \text{for capacitor-branches} \tag{2-3.8}$$

$$\begin{vmatrix} R_1, & 0 \\ 0, & G_2 \end{vmatrix} \begin{vmatrix} i_R \\ v_G \end{vmatrix} = \begin{vmatrix} v_R \\ i_G \end{vmatrix}, \quad \text{for resistor-branches}$$

$$\begin{vmatrix} L_1, & 0 \\ 0, & L_2 \end{vmatrix} \frac{d}{dt} \begin{vmatrix} i_L \\ i_\Gamma \end{vmatrix} = \begin{vmatrix} v_L \\ v_\Gamma \end{vmatrix}, \quad \text{for inductor-branches,}$$

where the matrices C_1, C_2, R_1, G_2, L_1 and L_2 are diagonal.

From the Kirchhoff's voltage and current laws, we obtain

$$\begin{vmatrix} v_S \\ v_R \\ v_L \end{vmatrix} - \begin{vmatrix} e_S \\ e_R \\ e_L \end{vmatrix} = - \begin{vmatrix} F_{SC}, & 0, & 0 \\ F_{RC}, & F_{RG}, & 0 \\ F_{LC}, & F_{LG}, & F_{L\Gamma} \end{vmatrix} \begin{vmatrix} v_C \\ v_G \\ v_\Gamma \end{vmatrix} \tag{2-3.9}$$

$$\begin{vmatrix} i_C \\ i_G \\ i_\Gamma \end{vmatrix} - \begin{vmatrix} j_C \\ j_G \\ j_\Gamma \end{vmatrix} = \begin{vmatrix} F'_{SC}, & F'_{RC}, & F'_{LC} \\ 0, & F'_{RG}, & F'_{LG} \\ 0, & 0, & F'_{L\Gamma} \end{vmatrix} \begin{vmatrix} i_S \\ i_R \\ i_L \end{vmatrix}, \tag{2-3.10}$$

where the loop source voltage vector e and the cut-set source current vector j are partitioned (for example, the k-th component of e_S is the algebraic sum of source voltages appearing in the *loop(*the k-th link-capacitor branch*)*).

Eliminating voltage and current vectors except v_C and i_L from Eq.(2-3.8), (2-3.9) and (2-3.10), we obtain a differential state equation as

$$\dot{x} = Ax + Bu, \tag{2-3.11}$$

where $x = \begin{vmatrix} v_C \\ i_L \end{vmatrix}$ $\quad u = \begin{vmatrix} j_C \\ j_G \\ j_\Gamma \\ e_S \\ e_R \\ e_L \end{vmatrix}$ (2-3.12)

$$A = \begin{vmatrix} \boldsymbol{C}^{-1}, & 0 \\ 0, & \boldsymbol{L}^{-1} \end{vmatrix} \begin{vmatrix} -\boldsymbol{y}, & H \\ -H', & -\boldsymbol{z} \end{vmatrix}$$

$$B = \begin{vmatrix} \boldsymbol{C}^{-1}, & 0 \\ 0 & \boldsymbol{L}^{-1} \end{vmatrix} \begin{vmatrix} 1, F'_{RC}\boldsymbol{R}^{-1}F_{RG}G_2^{-1}, & 0 & , F'_{SC}C_1\frac{d}{dt}, & F'_{RC}\boldsymbol{R}^{-1} & ,0 \\ 0 \quad -F_{LG}\boldsymbol{g}^{-1} & , -F_{L\Gamma}L_2\frac{d}{dt}, & 0 \quad , & -F_{LG}\boldsymbol{g}^{-1}F'_{RG}R_1 & ,1 \end{vmatrix}$$

$\boldsymbol{y} = F'_{RC}\boldsymbol{R}^{-1}F_{RC}$ $\qquad \boldsymbol{z} = F_{LG}\boldsymbol{g}^{-1}F'_{LG}$

$H = F'_{LC} - F_{RC}\boldsymbol{R}^{-1}F_{RG}G_2^{-1}F'_{LG}$

$\boldsymbol{C} = C_2 + F'_{SC}C_1F_{SC}$ $\qquad \boldsymbol{L} = L_1 + F_{L\Gamma}L_2F'_{L\Gamma}$

$\boldsymbol{R} = R_1 + F_{RG}G_2^{-1}F'_{RG}$ $\qquad \boldsymbol{g} = G_2 + F'_{RG}R_1^{-1}F_{RG}$

Property 2-3.1

The matrices $\boldsymbol{C}$, $\boldsymbol{L}$, $\boldsymbol{R}$ and $\boldsymbol{g}$ are positive definite.

Proof: Consider the matrix $\boldsymbol{C}$. From Eq.(2-3.12), it is rewritten as

$$\boldsymbol{C}=\left|-F'_{SC},1\right|\begin{vmatrix} C_1, & 0 \\ 0 & ,C_2 \end{vmatrix}\begin{vmatrix} -F_{SC} \\ 1 \end{vmatrix}. \qquad (2\text{-}3.13)$$

Then, the matrix $\boldsymbol{C}$ may be positive semi-definite. The first matrix of the right hand side of Eq.(2-3.13) is a fundamental cut-set matrix of $G\{R,L\}$ and the last matrix is a transposed matrix of the fundamental cut-set matrix. Therefore the determinant of $\boldsymbol{C}$ is given by

$$|\boldsymbol{C}|=\sum_{\text{all tree}}(\text{tree-capacitance product in } G\{R,L\}).$$

Since $|\boldsymbol{C}|\neq 0$ from this expansion, the matrix $\boldsymbol{C}$ is positive definite. Since the matrices $\boldsymbol{L}$, $\boldsymbol{R}$ and $\boldsymbol{g}$ are of the same form as $\boldsymbol{C}$, they are positive definite in similar way.

Q.E.D

Nonsigular $\boldsymbol{R}$ and $\boldsymbol{g}$ guarantee the existence of a hybrid matrix, whose rows correspond to the currents through the capacitors and the voltages across the inductors, and whose columns correspond to the voltages across the capacitors and the currents through the inductors. Nonsingular $\boldsymbol{C}$ and $\boldsymbol{L}$ guarantee that the state equation (2-3.11) has a unique solution. The above discussions are essential to deriving state equations.

Property 2-3.2

Let $h(i,j)$ denote an (i,j) element of H, then the absolute value $|h(i,j)|$ is not more than 1.

Property 2-3.2 is a result of the non-amplification property of a resistive network. Here another proof is given.

Proof: In order to obtain the value of $h(i,j)$, consider the graph $G_{ij}=G$[all capacitor-branches except C_i; all inductor branches except L_j}, where C_i and L_j are twig-capacitor and link-inductor corresponding to $h(i,j)$. The graph G_{ij} may be separable, that is, it may have more than one non-separable component. When C_i is not in the component where L_j is, $h(i,j)=0$. When C_i and L_j are in one component, consider the possibility of a normal tree in G_{ij} such that C_i is not in the $loop(L_j)$. But such a normal tree can or cannot be chosen. Consider the case where such a tree does not exist. Then, if both $f_{RC}\{i\}$ and $f_{LG}(j)$ are nonzero, the elements of F_{RG} corresponding to nonzero $f_{RC}\{i\}$ and $f_{LG}(j)$ are zero. For if they were nonzero, a normal tree with the above-mentioned property could have existed. Then we obtain

$$f'_{RC}(i)\mathbb{R}^{-1}F_{RG}G_2^{-1}f'_{LG}\{j\}=0. \qquad (2\text{-}3.14)$$

Eq.(2-3.14) holds evidently in the case where $f_{RC}\{i\}$ or $f_{LG}(j)$ is zero. Since C_i is in the $loop(L_j)$, $|h(i,j)|=1$ in this case.

Consider the case where there exists a normal tree for which C_i is not in the $loop(L_j)$. Let N_{ij} denote the corresponding 2-ports of G_{ij} as shown in Fig.2-3.1. Let $T_{2ij,i'j'}$ denote a 2-tree, where the nodes i and j are in one connected part and the nodes i' and j' are in the other connected part. Let $W_{ij,i'j'}$ denote the sum of 2-tree admittance products corresponding to 2-trees $T_{2ij,i'j'}$. The relation between voltages and currents of the 2-ports N_{ij} is assumed to be

$$\begin{vmatrix} v_i \\ v_j \end{vmatrix} = \begin{vmatrix} r_{ii}, & r_{ij} \\ r_{ji}, & r_{jj} \end{vmatrix} \begin{vmatrix} i_i \\ i_j \end{vmatrix} \qquad (2\text{-}3.15)$$

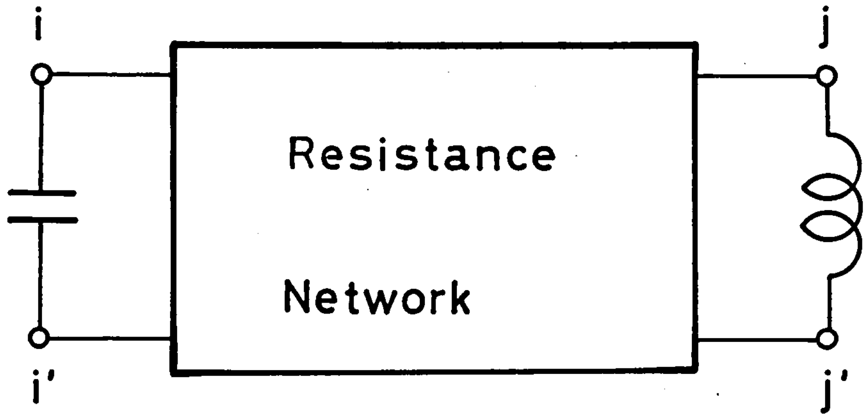


Fig.2-3.1 The corresponding 2-ports of G_{ij}.

From Eq.(2-3.15), the element $h(i,j)$ is described as

$$h(i,j)=-r_{ij}/r_{jj} \qquad (2\text{-}3.16)$$

The topological formula[4] leads

$$r_{ij}=\{W_{ij,i'j'}-W_{ij',i'j}\}/V(Y)$$

$$r_{jj}=W_{j,j'}/V(Y),$$

where $V(Y)$ denote the sum of tree admittance products and $W_{j,j'}$, the sum of tree admittance products corresponding to trees $T_{2j,j'}$ where the node j is in one connected part and the node j' in the other connected part. Since

$$W_{j,j'}=W_{ij,i'j'}+W_{ij',i'j}+W_{ii'j,j'}+W_{ii'j',j},$$

we obtain

$$|r_{ij}|<|r_{jj}| \qquad |h(i,j)|<1.$$

Consequently, we obtain

$$|h(i,j)|\leq 1, \qquad (2\text{-}3.17)$$

where the equal sign holds if there is no normal tree for which the branch C_i is not in the $loop(L_j)$.

Q.E.D

Property 2-3.2 does not hold in RLCT, RCG and active networks.

Property 2-3.3[1]

If the derivatives of input functions appear in the state equations, there exists at least one loop which consists of voltage source branches and capacitor branches only, or at least one cut-set which consists of current source branches and inductor branches only in G^*.

Property 2-3.4[5]

If there exists at least one cut-set which consists of capacitor-branches only, or at least one loop which consists of inductor-branches only in G, the matrix A of the state equation is singular.

CHAPER 3 RLCT NETWORK(1)

3-1 INTRODUCTION

A network considering in this chapter contains, as the network-elements, only resistors, inductors, capacitors, ideal-transformers and independent sources. Furthermore it is assumed to satisfy Assumption A in Chapter 2. The network is called an RLCT network N^*.

Several papers on the solvability of RLCT networks have been published[6][7][8]. A necessary condition for the solvability in a graphical sence has been given by [6]. Different from the case of RLC networks, however, solvability conditions depend not only on network-topology, but also on network-element values. A necessary and sufficient condition for the unique solvability is that the coefficient matrix of the network equation is nonsingular. In this chapter, a necessary and sufficient condition which depends on network-topology and turn-ratios of the ideal-transformers is given.

An explicit form of state equations of RLCT networks is given in [6], however, the process of deriving them is not discussed.

In Section 3-2,the magnetic and the electric properties of the multiwinding ideal-transformers are studied.

In Section 3-3, the problems on solvability and the order of complexity are studied.

In Section 3-4, the explicit forms of state equations are formulated.

3-2 MULTIWINDING IDEAL-TRANSFORMER

Consider an ideal-transformer five-port as shown in Fig.3-2.1, where two of the ports are conected to two inductors and the rest are free. Let v and i denote the voltage- and current-vectors of free ports. Then we obtain

$$v=K'\Lambda K\frac{d}{dt}i\triangleq M\frac{d}{dt}i, \qquad (3\text{-}2.1)$$

where $K=\begin{vmatrix} k_{11}, k_{12}, k_{13} \\ k_{21}, k_{22}, k_{23} \end{vmatrix} \quad \Lambda=\begin{vmatrix} L_1, & 0 \\ 0 & , L_2 \end{vmatrix}$.

The matrix M may be considered to be an inductance matrix of a mutual coupling inductor.

Consider a mutual coupling inductor whose inductance matrix is represented by an $n\times n$ matrix of the rank r, which is denoted by M. The matrix M is known to be symetric and positive semi-definite, therefore it is represented by

$$M=K'\Lambda K, \qquad (3\text{-}2.2)$$

where Λ is an $r\times r$ positive diagonal matrix. The matrix K is an $r\times n$ matrix whose elements $k(i,j)=0$, if $i>j$ and $k(i,i)=1$.

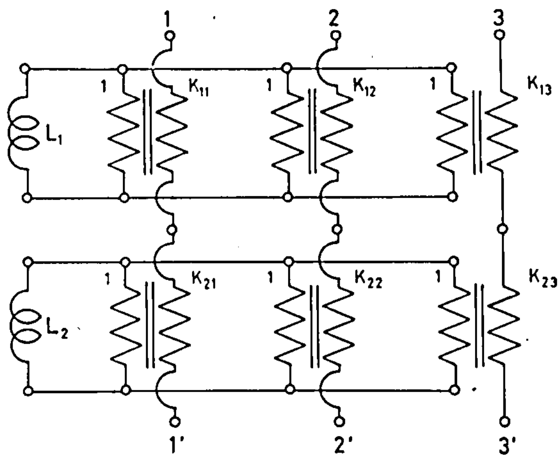


Fig.3-2.1 Example of ideal-transformer five-port.

Consequently a mutual coupling inductor is equivalent to an ideal-transformer $n+r$-port, where r port are connected to r inductors and n ports are free. For example a mutual coupling inductor which is characterized by 3×3 inductance matrix of rank 2 is equivalent to a network shown in Fig.3-3.2.

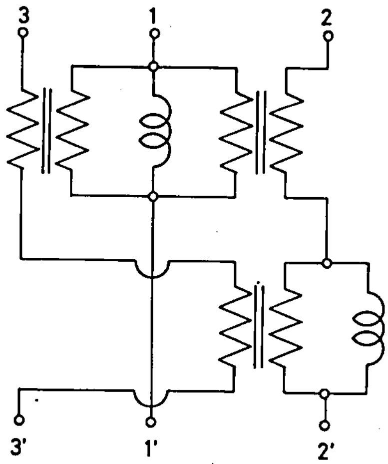


Fig.3-2.2 Equivalent network of mutual coupling inductor characterized by 3×3 inductance matrix of rank 2.

Therefore an RLCT network may be studied in stead of a general passive reciprocal network with mutual coupling inductors.

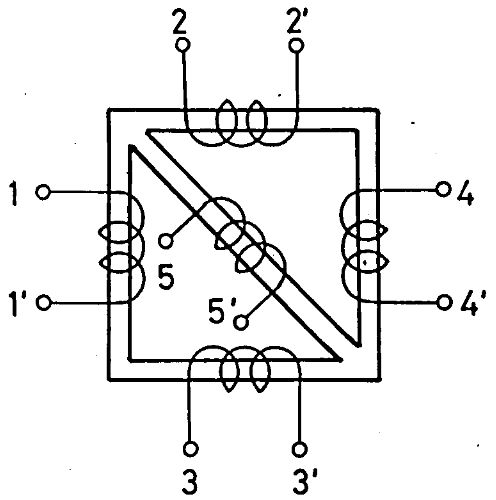
Consider the electric and magnetic properties of multiwinding ideal-transformers. The magnetic property of multiwinding ideal-transformers can be characterized by magnetic graphs. For example the magnetic graph of the multiwinding ideal-transformer shown in Fig.3-2.3(a) is that in Fig.3-2.3(b).

The magnetic graphs represent the magnetic circuits. Since ideal-transformers can be regarded as unity coupled inductors of infinite inductance, we have

$$Q_M \psi = 0$$

$$B_M \xi = 0, \qquad (3-2.3)$$

where ψ and ξ are the branch-flux and the branch-magnetomotive force vectors, respectively. The matrices Q_M and B_M are fundamental cut-set- and the fundamental loop matrices derived from the



(a) (b)

Fig.3-2.3 (a) Five winding ideal-transformer. (b) Its magnetic graph.

magnetic graph *M*. Let v_I and i_I denote the port-voltage- and the port-current vectors of ideal transformers. Let K denote the diagonal matrix that specifies turn-ratios of ideal-transformers. Then we obtain

$$v_I = K\frac{d}{dt}\psi \quad \text{or} \quad \frac{d}{dt}\psi = K^{-1} v_I$$

$$\xi = K i_I$$

$$K = diag.\{k_1, \cdots, k_n\}. \tag{3-2.4}$$

Eq.(3-2.3) and (3-2.4) lead us to

$$Q_W v_I = 0 \qquad Q_W \triangleq Q_M K^{-1}$$

$$B_W i_I = 0 \qquad B_W \triangleq B_M K \tag{3-2.5}$$

Since the matrices B_M and Q_M are ,respectively, written as

$$B_M = [1, F_M] \qquad Q_M = [-F_M', 1],$$

where F_M is the characteristic part of the fundamental loop matrix derived from the magnetic graph *M*. Eq.(3-2.5) becomes

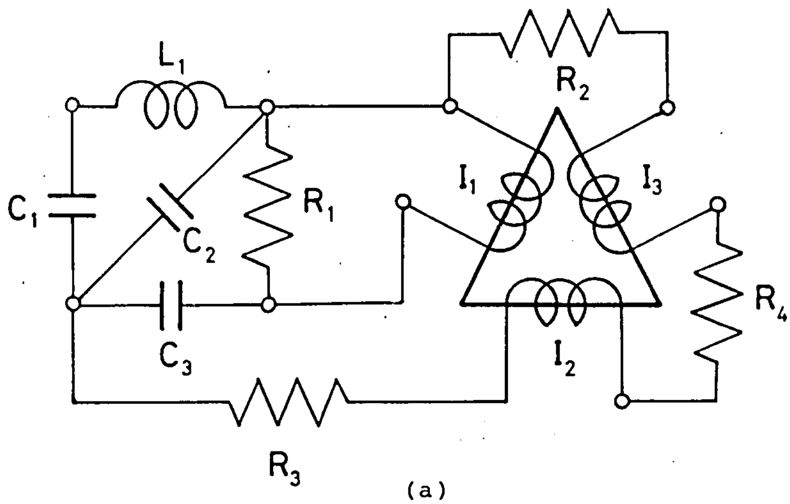
$$v_t = K_t F_M' K_l^{-1} v_l \triangleq M' v_l$$

$$i_l = -K_l^{-1} F_M K_t i_t \triangleq -M i_t, \qquad (3\text{-}2.6)$$

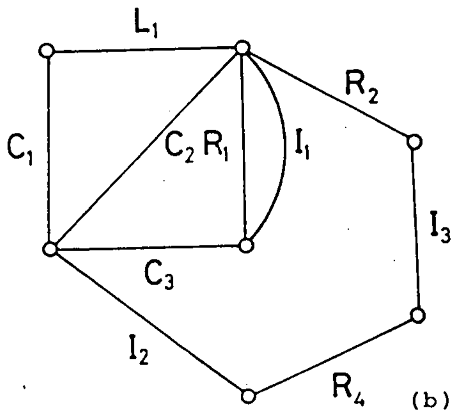
where $\begin{vmatrix} v_l \\ v_t \end{vmatrix} = v_I$ $\begin{vmatrix} i_l \\ i_t \end{vmatrix} = i_I$ $\begin{vmatrix} K_l, & 0 \\ 0, & K_t \end{vmatrix} = K$

The subscripts l and t denote links and twigs, respectively, of the magnetic graph M.

Consider the electric property of an RLCT network. It can be characterized by an electric graph G which is obtained from an RLCT network by replacing each of two terminal elements and each port of ideal-transformers with a branch. For example, the electric and magnetic graphs of the RLCT network shown in Fig.3-2.4(a) are in Fig.3-2.4(b) and (c), respectively.

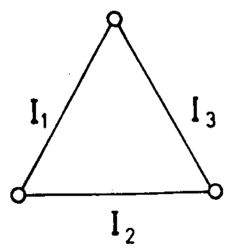


(a)



(b)

Fig.3-2.4
(a) RLCT network
(b) its electric graph
(c) its magnetic graph.



(c).

In electric and magnetic graphs (G and M), every branch in M corresponds to a branch in G. However only every transformer branch in G has one-to-one correspondence with that of M.

we define

Definition[5] 3-2.1

A proper tree T_p of an RLCT network is a set of branches which forms a tree in its electric graph G. Besides the transformer branches in the set form a tree in its magnetic graph M.

Consider an algorithm of selecting a proper tree of RLCT networks. Let S_{G1}"S_{G2}" denote a set of all branches in G_1"G_2".

Definition 3-2.2

A common tree is a tree in two graphs G_1 and G_2 where each branch in G_1 has a one-to-one correspondence with a branch in G_2, *i.e.* $S_{G1}=S_{G2}\triangleq S_G$.

Selecting common trees have been given in [10] and [11]. They may be summerized as follows. Let T_1"T_2" denote a tree of G_1"G_2". Then we obtain

$$T_1\cup\bar{T}_2=\bar{T}_2\cup(T_1\cap T_2). \tag{3-2.7}$$

Since $\bar{T}_2\cap(T_1\cap T_2)=\phi$, we obtain

$$|T_1\cup\bar{T}_2|=|\bar{T}_2|+|T_1\cap T_2|. \tag{3-2.8}$$

Since $|T_1\cap T_2|\leqq|T_1|$ (or $|T_2|$) and $|\bar{T}_2|$ is constant, if and only if $|T_1\cap T_2|$ is maximized, $|T_1\cup\bar{T}_2|$ is maximized. Since T_1 is identical with T_2 if and only if $|T_1\cup\bar{T}_2|=|S_G|$, the algorithm is one of maximizing $|T_1\cup\bar{T}_2|$, instead of $|T_1\cap T_2|$.

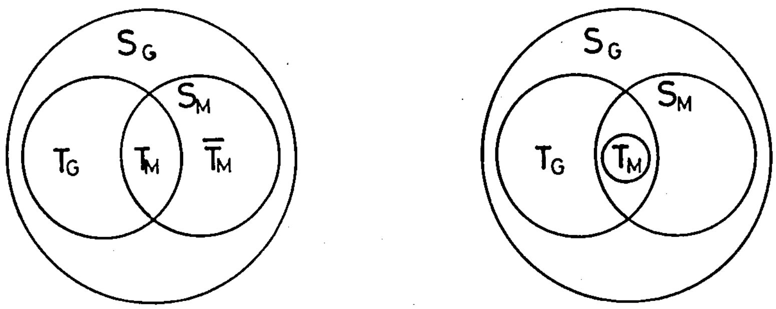
Modifying the above common tree algorithm, we obtain an algorithm of a proper tree in RLCT networks.

Let S_G"S_M" denote a set of all branches in G"M". The set S_M is a subset of S_G, that is, $S_G \; S_M$. Let T_G"T_M" denote a tree of G"M". Corresponding to Eq.(3-2.7) and (3-2.8), the following equations hold.

$$\bar{T}_G \cup T_M = \bar{T}_G \cup (T_G \cap T_M) \tag{3-2.9}$$

$$\begin{aligned} |\bar{T}_G \cup T_M| &= |\bar{T}_G| + |T_G \cap T_M| \\ &\leqq \mu(G) + r(M). \end{aligned} \tag{3-2.10}$$

By use of the common tree algorithm, trees T_G and T_M such that $|\bar{T}_G \cup T_M|$ is maximum can be chosen. If T_G is a proper tree, it is necessary that $\bar{T}_G \cap T_M = \phi$ and $T_G \; \bar{T}_M = \phi$ as shown in Fig.3-2.5(a). In general, $T_G \cap \bar{T}_M \neq \phi$, even if $|T_G \cup T_M|$ is maximum and $\bar{T}_G \cap T_M = \phi$.(as shown in Fig.3-2.5(b).) Therefore $|\bar{T}_G \cup T_M|$ is maximized by the common tree algorithm, and then if $\bar{T}_G \cap T_M = \phi$, $|T_G \cup \bar{T}_M|$ is maximized by its dual algorithm. If $T_G \cup \bar{T}_M = \phi$, T_G is a proper tree.



(a) (b)

Fig.3-2.5 Venn's diagrams. (a) T_G is a proper tree.
(b) $|T_G \cup T_M|$ is maximum and $\bar{T}_G \cap T_M = \phi$.

Let us obtain a proper tree of an RLCT network whose electric and magnetic graphs are given in Fig.3-2.6 Assume that T_G and T_M

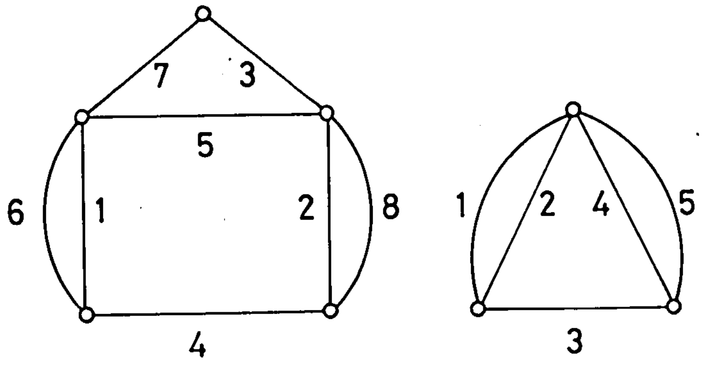


Fig. 3-2.5 Example. (a) Electric graph G.
(b) Magnetic graph M.

are initially as $T_G=\{3,4,5,6\}$ and $T_M=\{1,3\}$. By the common tree algorithm, T_G and T_M such that $|\bar{T}_G \cup T_M|$ is a maximum are obtained as $T_G=\{2,3,4,5\}$ and $T_M=\{1,3\}$. From these, $|T_G \cup \bar{T}_M|$ is maximized, and then a proper tree is obtained as $T_p=\{2,3,6,7\}$, where $T_G=T_p$ and $T_M=\{2,3\}$.

Let us define a proper tree giving the order of complexity.

Definition[6] 3-2.3

A maximum proper tree T_{pmax} of an RLCT network is defined to be a proper tree having maximum $(|T_p(C)|+|T_p(L)|)$.

3-3 NETWORK EQUATION AND SOLVABILITY

A network equation of an RLCT network is given by

$$\theta(p)\begin{vmatrix} i \\ v \end{vmatrix} \triangleq \begin{array}{cccc} (\Sigma) & (M) & (\Sigma) & (M) \\ \left|\begin{matrix} Q_\Sigma, & Q_M, & 0, & 0 \\ 0, & 0, & B_\Sigma, & B_M \\ Z, & 0, & -Y, & 0 \\ 0, & B_W, & 0, & 0 \\ 0, & 0, & 0, & Q_W \end{matrix}\right| \end{array} \begin{vmatrix} i_\Sigma \\ i_M \\ v_\Sigma \\ v_M \end{vmatrix} = \begin{vmatrix} j \\ e \\ 0 \\ 0 \end{vmatrix}, \qquad (3\text{-}3.1)$$

where the subscript M and Σ denote the ideal-transformers and the network-elements except the ideal-transformers, respectively. The matrix $Q\triangleq|Q_\Sigma, Q_M|$ "$B\triangleq|B_\Sigma, B_M|$" is a basic cut-set matrix"a basic loop matrix" in the electric graph.

The next theorem is given in [6].

Theorem 3-3.1[6]

A necessary condition for an RLCT network to have a unique solution is that there exists at least one proper tree in its electric graph.

The proof of theorem 3-3.1 is not given in [6] and therefore we give one here.

(proof) By the Laplace expansion of $det|\theta(p)|$, we obtain

$$det|\theta(p)| = \Sigma sgn(S_\Sigma)\cdot sgn(S_I)|[Q_\Sigma, Q_M]|_{S_\Sigma \cup S_I}|[B_\Sigma, B_M]|_{\bar{S}_\Sigma \cup \bar{S}_I}$$

$$\times|Z;-Y|_{\bar{S}_\Sigma : S_\Sigma}|B_W|_{\bar{S}_I}|Q_W|_{S_I}, \qquad (3\text{-}3.2)$$

where

S_Σ: A subset of $S_G - S_M$

S_I: A subset of S_M

$\bar{S}_\Sigma$: $S_G - S_M - S_I$

$\bar{S}_I$: $S_M - S_I$

$|[Q_\Sigma,Q_M]|_{S_\Sigma\cup S_I}$ "$|[B_\Sigma,B_M]|_{\bar{S}_\Sigma\cup\bar{S}_I}$" = The determinant of the submatrix which is composed of the columns of $[Q_\Sigma,Q_M]$"$[B_\Sigma,B_M]$" corresponding to $S_\Sigma\cup S_I$"$\bar{S}_\Sigma\cup\bar{S}_I$".

$|Z;-Y|_{\bar{S}_\Sigma:S_\Sigma}$ = The determinant of the diagonal matrix whose elements of Z corresponding to $\bar{S}_\Sigma$ and those of -Y to S_Σ.

since $det|\theta(p)|\neq 0$ if the network has a unique solution, there exist S_Σ and S_I such that

$$|[Q_\Sigma,Q_M]|_{S_\Sigma\cup S_I}|[B_\Sigma,B_M]|_{\bar{S}_\Sigma\cup\bar{S}_I}|B_W|_{\bar{S}_I}|Q_W|_{S_I}\neq 0,$$

that is, $S_\Sigma\cup S_I$ is a tree in G and S_I is a tree in M. Consequently $S_\Sigma\cup S_I$ is a proper tree.

Q.E.D.

Theorem 3-3.2[6]

The order of complexity of an RLCT network is not greater than the sum of the number of capacitors in any maximum proper tree and the number of inductors in its corresponding cotree.

Above two theorems are obtained in considerations of network-topology. However, solvability of an RLCT network, that is, whether it has a unique solution, depends upon not only network-topology but also network-element-values.

Consider how the network element-values affect network solvability. Assuming that a proper tree T_p can be obtained, we have the fundamental loop and cut-set matrices as

$$B=\begin{vmatrix} F_{LT}, 1, F_{Lt}, 0 \\ F_{lT}, 0, F_{lt}, 1 \end{vmatrix} \qquad (3\text{-}3.3)$$

$$Q=\begin{vmatrix} 1, -F'_{LT}, 0, -F'_{lT} \\ 0, -F'_{Lt}, 1, -F'_{lt} \end{vmatrix},$$

where the subscript L"T" denotes the links"twigs" in Σ. The subscript l"t" denotes the links" twigs" in M.

Let $T_p(\Sigma)$"$T_p(M)$" denote the subset of T_p excluding the ideal-transformer-branches"$T_p - T_p(\Sigma)$". Then the coefficient matrix may be written as

$$
\begin{array}{c}
\begin{array}{cccccccc}
T_p(\Sigma) & \bar{T}_p(\Sigma) & T_p(M) & \bar{T}_p(M) & T_p(\Sigma) & \bar{T}_p(\Sigma) & T_p(M) & \bar{T}_p(M)
\end{array}\\
\theta_{0s}(p)=\begin{vmatrix}
1, & -F'_{LT}, & 0, & -F'_{lT}, & & & 0 & \\
0, & -F'_{Lt}, & 1, & -F'_{lt}, & & & & \\
 & & & & F_{LT}, & 1, & F_{Lt}, & 0\\
 & 0 & & & F_{lT}, & 0, & F_{lt}, & 1\\
Z_T, & 0, & & & -Y_T, & 0, & & \\
0, & Z_L, & 0 & & 0, & -Y_L & 0 & \\
0, & 0, & F_M K_t, & K_l, & 0, & 0, & 0, & 0\\
0, & 0, & 0, & 0, & 0, & 0, & K_t^{-1}, & -F_M K_l^{-1}
\end{vmatrix},
\end{array}
\tag{3-3.4}
$$

where the matrix Z_L"Z_T" is a diagonal matrix whose diagonal elements are those of Z corresponding to the links"twigs" of T_p, and the matrix Y_L"Y_T" is a diagonal matrix whose diagonal elements are those of Y corresponding to the links"twigs" of T_p.

Consider the term in the Laplace expansion of $det|\theta_{0s}(p)|$ corresponding to $T_p(\Sigma)$. It is rewritten by

$$
\begin{aligned}
|\theta_{0s}(p)|_{T_p(\Sigma)} &= a|Z_L||Y_T||K_l + F_M K_t F'_{lt}||K_t^{-1} + F'_M K_l^{-1} F_{lt}| \\
&\triangleq a|Z_L||Y_T|a^0_{T_p(\Sigma)},
\end{aligned}
\tag{3-3.5}
$$

where $a=\pm 1$.

consider the term in $|\theta_{0s}(p)|$ corresponding to another proper tree T_{p1}. By exchanging the rows and columns of $\theta_{0s}(p)$, it becomes

$$
\theta_1(p)=\begin{array}{c}
\begin{array}{cccccccc} T_{p1}(\Sigma) & \bar{T}_{p1}(\Sigma) & T_{p1}(M) & \bar{T}_{p1}(M) & T_{p1}(\Sigma) & \bar{T}_{p1}(\Sigma) & T_{p1}(M) & \bar{T}_{p1}(M) \end{array} \\
\begin{vmatrix}
{}_1Q_1 & {}_1Q_2 & {}_1Q_3 & {}_1Q_4 & & & & \\
{}_1Q_5 & {}_1Q_6 & {}_1Q_7 & {}_1Q_8 & & 0 & & \\
 & & & & {}_1B_1 & {}_1B_2 & {}_1B_3 & {}_1B_4 \\
 & 0 & & & {}_1B_5 & {}_1B_6 & {}_1B_7 & {}_1B_8 \\
{}_1Z_T & 0 & & & -{}_1Y_T & 0 & & \\
0 & {}_1Z_L & & 0 & 0 & -{}_1Y_L & & 0 \\
 & 0 & {}_1B_{Wt} & {}_1B_{Wl} & & 0 & & \\
 & 0 & & & & 0 & {}_1Q_{Wt} & {}_1Q_{Wl}
\end{vmatrix}
\end{array},
\qquad (3\text{-}3.6)
$$

where the matrix ${}_1Z_L$"${}_1Z_T$" is a diagonal matrix whose diagonal elements are those of Z corresponding to the links"twigs" of T_{p1}. The matrix ${}_1Y_L$"${}_1Y_T$" is a diagonal matrix whose diagonal elements are those of Y corresponding to the links"twigs" of T_{p1}. The matrix ${}_1Q$"${}_1B$" is obtained from Q"B" by exchanging columns, and

$$
{}_1Q=\begin{vmatrix} {}_1Q_1, {}_1Q_2, {}_1Q_3, {}_1Q_4 \\ {}_1Q_5, {}_1Q_6, {}_1Q_7, {}_1Q_8 \end{vmatrix}
$$

$$
{}_1B=\begin{vmatrix} {}_1B_1, {}_1B_2, {}_1B_3, {}_1B_4 \\ {}_1B_5, {}_1B_6, {}_1B_7, {}_1B_4 \end{vmatrix}.
$$

Assume that T_{p1} is obtained from T_p by m_1 tree-transformations among the branches of Σ and n_1 tree-transformations among the branches of M. The matrix $\theta_1(p)$ is obtained from $\theta_{0s}(p)$ by m_1+n_2 row-exchanges and $2m_1+n_2$ column-exchanges.

By the elementary transformation, $\theta_1(p)$ is transformed to $\theta_{1s}(p)$ as

$$
\begin{array}{cccccccc}
T_{p1}(\Sigma) & \bar{T}_{p1}(\Sigma) & T_{p1}(M) & \bar{T}_{p1}(M) & T_{p1}(\Sigma) & \bar{T}_{p1}(\Sigma) & T_{p1}(M) & \bar{T}_{p1}(M)
\end{array}
$$

$$
\theta_{1s}(p)=\left|\begin{array}{cccccccc}
1 & -{}_1F'_{LT} & 0 & -{}_1F'_{1T} & & & & \\
0 & -{}_1F'_{Lt} & 1 & -{}_1F'_{1t} & & 0 & & \\
 & & & & {}_1F_{LT} & 1 & {}_1F_{Lt} & 0 \\
 & 0 & & & {}_1F_{1T} & 0 & {}_1F_{1t} & 1 \\
{}_1Z_T & 0 & & & -{}_1Y_T & 0 & & \\
0 & {}_1Z_L & & 0 & 0 & -{}_1Y_L & & 0 \\
 & 0 & {}_1F_M\cdot{}_1K_1 & {}_1K_t & & 0 & & \\
 & 0 & & & & 0 & {}_1K_1^{-1} & {}_1F'_M\cdot{}_1K_t^{-1}
\end{array}\right|
\qquad (3\text{-}3.7)
$$

where ${}_1F_{\circ\circ}$ denote the submatrices of the characteristic part of the fundamental loop matrix for the tree T_{p1} (associated with the subscripts). The matrix ${}_1K_1$"${}_1K_t$" is a diagonal matrix whose diagonal elements are those of K corresponding to the links"twigs" of $T_{p1}(M)$.

The term in the Laplace expansion of $det|\theta_{1s}(p)|$ corresponding to $T_{p1}(\Sigma)$ is written as

$$
\begin{aligned}
|\theta_{1s}(p)|_{T_{p1}(\Sigma)} &= a|{}_1Z_L||{}_1Y_T||{}_1K_1+{}_1F_M\cdot{}_1K_t\cdot{}_1F'_{1t}| \\
&\qquad \times|{}_1K_t^{-1}+{}_1F'_M\cdot{}_1K_1^{-1}\cdot{}_1F_{1t}| \\
&\triangleq a|{}_1Z_L||{}_1Y_T|a^1_{T_{p1}(\Sigma)}. \qquad (3\text{-}3.8)
\end{aligned}
$$

Let us compare $|\theta_{1s}(p)|_{T_{p1}(\Sigma)}$ with $|\theta_{0s}(p)|_{T_{p1}(\Sigma)}$ which is the term in the Laplace expansion of $det|\theta_{1s}(p)|$ corresponding to $T_{p1}(\Sigma)$. From Lemma 2-2.1, we obtain

$$
\begin{aligned}
|\theta_{0s}(p)|_{T_{p1}(\Sigma)} &= (-1)^{3m_1+3n_1}\cdot(-1)^{m_1+n_1}|\theta_{1s}(p)|_{T_{p1}(\Sigma)} \\
&= |\theta_{1s}(p)|_{T_{p1}(\Sigma)}. \qquad (3\text{-}3.9)
\end{aligned}
$$

The term $a^1_{T_{p1}(\Sigma)}$ is written as

$$a^1_{T_{p1}(\Sigma)} = |{}_1K_1 + {}_1F_M \cdot {}_1K_t \cdot {}_1F'_{1t}|\,|{}_1K_t^{-1} + {}_1F'_M \cdot {}_1K_1^{-1} \cdot {}_1F_{1t}|$$
$$= |{}_1K_1|\,|{}_1K_t|^{-1}\,|1 + {}_1K_1^{-1} \cdot {}_1F_M \cdot {}_1K_t \cdot {}_1F'_{1t}|$$
$$\times |1 + {}_1F'_{1t} \cdot {}_1K_1^{-1} \cdot {}_1F_M \cdot {}_1K_t| \qquad (3\text{-}3.10)$$

From Lemma A-3.1, we obtain

$$a^1_{T_{p1}(\Sigma)} = |{}_1K_1|^{-1}\,|{}_1K_t|^{-1}\,|{}_1K_1 + {}_1F_M \cdot {}_1K_t \cdot {}_1F'_{1t}|^2$$
$$= |K|^{-1} g_1^2(k_1, k_2, \cdots\cdots, k_{|M|}), \qquad (3\text{-}3.11)$$

where $g_1(k_1, k_2, \quad , k_{|M|}) = |{}_1K_1 + {}_1F_M \cdot {}_1K_t \cdot {}_1F'_{1t}|$, which is a polynominal of turn ratios $k_i \{i=1,2, \quad ,|M|\}$.

Only one proper tree T_{p1} obtained from T_p by tree-transformation is considered above, however, without loss of generality, Eq.(3-3.9) and (3-3.11) hold for any proper tree. Then we obtain

Lemma 3-3.1

If an RLCT network has at least one proper tree, the determinant of the coefficient matrix $\theta(p)$ in the network equations is given by

$$det|\theta(p)| = a \prod_{i=1}^{|M|} k_i^{-1} [\sum_{T_{pm}(\Sigma)} \{ \prod_{C_j} C_j p^{\,}_{T_{pm}(\Sigma)} \prod_{R_k} R_k{}_{\bar{T}_{pm}(\Sigma)} \prod_{L_l} L_l p_{\bar{T}_{pm}(\Sigma)}$$
$$\times g_m^2(k_1, k_2, \quad , k_{|M|}) \}], \qquad (3\text{-}3.12)$$

where $g_m(k_1, k_2, \cdots\cdots, k_{|M|})$ is a polynominal of k_i, which is specified by the m-th proper tree T_{pm}.

From Lemma 3-3.1, two theorems are obtained

Theorem 3-3.3

A necessary and sufficient condition for an RLCT network to have a unique solution is that there exists at least one proper tree such that $det|K_1 + F_M K_t F'_{1t}| \neq 0$.

Proof: From Eq.(3-3.12), $det|\theta(p)|\neq 0$ if and only if $g_m^2(k_1,k_2,\cdots,k_{|M|})\neq 0$. Since $g_m=|{}_m\mathrm{K}_1+{}_m\mathrm{F}_\mathrm{M}\cdot{}_m\mathrm{K}_t\cdot{}_m\mathrm{F}'_{1t}|$, then the theorem holds.

Q.E.D.

We have a sufficient condition depending on network topology only, that is,

Theorem 3-3.4

A sufficient condition for an RLCT network to have a unique solution is that there exists at least one proper tree T_p such that ther exists no other proper tree which contains $T_p(\Sigma)$.

Proof: For a proper tree satisfying the above condition, g_m is a monomial (not a polynominal). Then, $g_m\neq 0$.

Q.E.D.

Next, the theorems concerning the order of complexity are obtained.

Theorem 3-3.5

If there exists a maximum proper tree T_{pmax} for which $det|\mathrm{K}_1+\mathrm{F}_\mathrm{M}\mathrm{K}_t\mathrm{F}'_{1t}|\neq 0$, the order of complexity of an RLCT network is equal to $|T_{pmax}(C)+\bar{T}_{pmax}(L)|$.

Theorem 3-3.6

If there exists no other proper tree which contains a maximum proper tree $T_{pmax}(\Sigma)$, the order of complexity of an RLCT network is equal to $|T_{pmax}(C)+\bar{T}_{pmax}(L)|$.

Some examples of networks for Theorem 3-3.4 - 3-3.6 are shown in Section 3-5.

3-4 STATE EQUATION

In this section, a method of formulating state equations of RLCT networks for maximum proper trees is studied.

The characteristic part of the fundamental loop matrix for a maximum proper tree T_{pmax} of an RLCT network is assumed to be given as

$$\begin{vmatrix} F_{SC}, & 0, & 0, & F_{St} \\ F_{RC}, & F_{RG}, & 0, & F_{Rt} \\ F_{LC}, & F_{LG}, & F_{L\Gamma}, & F_{Lt} \\ F_{1C}, & F_{1G}, & F_{1\Gamma}, & F_{1t} \end{vmatrix}. \qquad (3\text{-}4.1)$$

From the matrix (3-3.1), Kirchhoff's voltage and current laws state,

$$\begin{aligned} -v_S+e_S &= F_{SC}v_C + F_{St}v_t && (a) \\ -v_R+e_R &= F_{RC}v_C+F_{RG}v_G + F_{Rt}v_t && (b) \\ -v_L+e_L &= F_{LC}v_C+F_{LG}v_G+F_{L\Gamma}v_\Gamma+F_{Lt}v_t && (c) \\ -v_1+e_1 &= F_{1C}v_C+F_{1G}v_G+F_{1\Gamma}v_\Gamma+F_{1t}v_t && (d) \end{aligned} \qquad (3\text{-}4.2)$$

$$\begin{aligned} i_C-j_C &= F'_{SC}i_S+F'_{RC}i_R+F'_{LC}i_L+F'_{1C}i_1 && (a) \\ i_G-j_G &= F'_{RG}i_R+F'_{LG}i_L+F'_{1G}i_1 && (b) \\ i_\Gamma-j_\Gamma &= F'_{L\Gamma}i_L+F'_{1\Gamma}i_1 && (c) \\ i_t-j_t &= F'_{St}i\ +F'_{Rt}i\ +F'_{Lt}i\ +F'_{1t}i_1, && (d) \end{aligned} \qquad (3\text{-}4.3)$$

where e∘"j∘" is a loop source voltage vector"a cut-set source current vector" associated with the subscripts.

From Eq.(3-2.6), (3-4.2d) and (3-4.3d), we obtain

$$-(K_l+F_{lt}K_tF'_M)K_l^{-1}v_l=F_{lC}v_C+F_{lG}v_G+F_{l\Gamma}v_\Gamma-e_l \tag{3-4.4}$$
$$-(K_t+F'_{lt}K_lF_M)K_t^{-1}i_t=F'_{St}i_S+F'_{Rt}i_R+F'_{Lt}i_L+j_t \tag{3-4.5}$$

If T_{pmax} satisfies the conditions of Theorem 3-3.5, that is, if the matrix $(K_l+F_{lt}K_tF'_M)$ is nonsingular, then the matrix $(K_t+F'_{lt}K_lF_M)$ is nonsingular from Lemma A-3.1. Then the vectors v_l, v_t, i_l and i_t can be eliminated from Eq.(3-4.2) and (3-4.3), and we obtain

$$-v_S=\{F_{SC}-F_{St}\widetilde{M}F_{lC}\}v_C-F_{St}\widetilde{M}F_{lG}v_G-F_{St}\widetilde{M}F_{l\Gamma}v_\Gamma-e_S+F_{St}\widetilde{M}e_l \tag{a}$$
$$-v_R=\{F_{RC}-F_{Rt}\widetilde{M}F_{lC}\}v_C+\{F_{RG}-F_{Rt}\widetilde{M}F_{lG}\}v_G-F_{Rt}\widetilde{M}F_{l\Gamma}v_\Gamma-e_R+F_{Rt}\widetilde{M}e_l \tag{b}$$
$$-v_L=\{F_{LC}-F_{Lt}\widetilde{M}F_{lC}\}v_C+\{F_{LG}-F_{Lt}\widetilde{M}F_{lG}\}v_G+\{F_{L\Gamma}-F_{Lt}\widetilde{M}F_{l\Gamma}\}v_\Gamma -e_L+F_{Lt}\widetilde{M}e_l \tag{c}$$
(3-4.6)

$$i_C=\{F'_{SC}-F'_{lC}\widetilde{M}F'_{St}\}i_S+\{F'_{RC}-F'_{lC}\widetilde{M}F'_{Rt}\}i_R+\{F'_{LC}-F'_{lC}\widetilde{M}F'_{Lt}\}i_L +j_C-F'_{lC}\widetilde{M}j_t \tag{a}$$
$$i_G=-F'_{lG}\widetilde{M}'F'_{St}i_S+\{F'_{RG}-F'_{lG}\widetilde{M}'F'_{Rt}\}i_R+\{F'_{LG}-F'_{lG}\widetilde{M}'F'_{Lt}\}i_L+j_G-F'_{lG}\widetilde{M}'j_t \tag{b}$$
$$i_\Gamma=-F'_{l\Gamma}\widetilde{M}'F'_{St}i_S-F'_{l\Gamma}\widetilde{M}'F'_{Rt}i_R+\{F'_{L\Gamma}-F'_{l\Gamma}\widetilde{M}'F'_{Lt}\}i_L+j_\Gamma-F'_{l\Gamma}\widetilde{M}'j_t, \tag{c}$$
(3-4.7)

where $\widetilde{M}=M'(1+F_{lt}M')^{-1}$.

If the matrix-products $F_{St}\widetilde{M}F_{lG}$, $F_{St}\widetilde{M}F_{l\Gamma}$, $F_{Rt}\widetilde{M}F_{l\Gamma}$, $F'_{lG}\widetilde{M}F'_{St}$, $F'_{l\Gamma}\widetilde{M}'F'_{St}$ and $F'_{l\Gamma}\widetilde{M}'F'_{Rt}$, in Eq.(3-4.6) and (3-4.7) are zero-matrices, Eq.(3-4.6) and (3-4.7) are of the same form as Eq.(2-3.9) and (2-3.10) in Chapter 2. Then a state equation of an RLCT network can be obtained by the same procedure as in Section 2-3.

Lemma 3-4.1

The matrix-products $F_{St}\widetilde{M}F_{lG}$, $F_{St}\widetilde{M}F_{l\Gamma}$, $F_{Rt}\widetilde{M}F_{l\Gamma}$, $F'_{lG}\widetilde{M}'F'_{St}$, $F'_{l\Gamma}M'F'_{St}$ and $F'_{l\Gamma}M'F'_{Rt}$ are all zero-matrices for any maximum proper tree.

Proof: Assume that a (p,q) element of $F_{St}\widetilde{M}F_{l\Gamma}$ is nonzero. Then there exist m and i such that

$$f_{St}(p,m)\widetilde{m}(m,i)f_{l\Gamma}(i,q)\neq 0. \tag{3-4.8}$$

If $f_{St}(p,m)\neq 0$, there exist a link-capacitor-branch (denoted by S_p) corresponding to the p-th row of F_{St}, $f_{St}(p)$, and a twig-transformer-branch (denoted by t_m) in the $loop(S_p)$ in the electric graph G. If $f_{l\Gamma}(i,q)\neq 0$, there exist a link-transformer-branch (denoted by l_i) corresponding to $f_{l\Gamma}(i)$ and a twig-inductor-branch (denoted by Γ_q) in the $loop(l_i)$ in G.

Examine the relation between $m(m,i)\neq 0$ and the network-topology. From Eq.(3-2.6), the matrix $\widetilde{M}$ is written as

$$\begin{aligned}\widetilde{M}&=M'(1+F_{lt}M')^{-1}\\ &=K_tF_M'K_l^{-1}(1+F_{lt}K_tF_M'K_l^{-1})^{-1}\\ &=K_tF_M'\{[1,F_{lt}]\begin{vmatrix}K_l, & 0\\ 0 & ,K_t\end{vmatrix}\begin{vmatrix}1\\ F_M'\end{vmatrix}\}^{-1}\\ &\triangleq K_tF_M'K^{-1},\end{aligned} \tag{3-4.9}$$

where

$$K=[1,F_{lt}]\begin{vmatrix}K_l, & 0\\ 0 & ,K_t\end{vmatrix}\begin{vmatrix}1\\ F_M'\end{vmatrix}. \tag{3-4.10}$$

The matrices $[1,F_{lt}](\triangleq B_I)$ and $[1,F_M](\triangleq B_M)$ are the fundamental loop matrices of the electric graph $G[\dot{T}_{pmax}(\Sigma);\bar{T}_{pmax}(\Sigma)\}$(denoted by G_I) and the magnetic graph M.

If $\widetilde{m}(m,i)\neq 0$, there exist j and k such that

$$k_t(m,j)f_M(k,j)K^{-1}(k,j)\neq 0 \tag{3-4.11}$$

If $k_t(m,j)\neq 0$, $m=j$, since K_t is a diagonal matrix. If $f_M(k,j)\neq 0$, there exist a twig-transformer-branch t_m in M, and a link-transformer-branch (denoted by l_k) in the *cut-set*(t_m) in M.

Examine the network topology when $k^{-1}(k,i)\neq 0$. The value $k^{-1}(k,i)$ is given by

$$k^{-1}(k,i)=Cof.K(i,k)/|K|. \qquad (3\text{-}4.12)$$

Therefore it is rewritten as

$$k^{-1}(k,i)=(-1)^{k+i}|B_I(i)KB'_M(k)|/|K|, \qquad (3\text{-}4.13)$$

where $B_I(i)$ is the matrix obtained from B_I by deleting the i-th row. The matrix $B_I(i)$ is the fundamental loop matrix of the graph $G_I\{l_i\}$. The matrix $B_M(k)$ is the fundamental loop matrix of the graph $M\{l_k\}$.

From the electro-magnetic point of view, the magnetic-motive force of l_i is zero since l_i is deleted in $G_I\{l_i\}$. Therefore the branch l_i in M may be considered to be contracted. The induced electro-motive force of l_k is zero since l_k is deleted in $M\{l_k\}$. Therefore the branch l_k in G_I may be considered to be contracted. Consequently, two graphes $G_I[l_k;l_i\}$ and $M[l_i;l_k\}$ may be considered for Eq.(3-4.13), which may be rewritten as

$$k^{-1}(k,i)=(-1)^{k+i}|B_{G_I[l_k;l_i\}}K\{i,k\}B'_{M[l_i;l_k\}}|/|K|, \qquad (3\text{-}4.14)$$

where $B_{G_I[l_k;l_i\}}$ "$B_{M[l_i;l_k\}}$" is the fundamental loop matrix of $G_I[l_k;l_i\}$ "$M[l_i;l_k\}$". The matrix $K\{i,k\}$ is a matrix obtained by removing the i-th row and the k-th column from K. That $k^{-1}(k,i)\neq 0$ leads to that there exists at least one proper tree in $G_I[l_k;l_i\}$ whose magnetic graph is $M[l_i;l_k\}$

Consequently, if $f_{St}(p,m)m(m,i)f_{l\Gamma}(i,q)\neq 0$, there exist (1) a twig-transformer-branch t_m in *loop*(S_p) and a link-transformer-branch l_i in *cut-set*(Γ_q) in $G[T_{pmax}(\Sigma)-\Gamma_q;\bar{T}_{pmax}(\Sigma)-S_p\}$, (2) a link-transformer-branch l_k in *cut-set*(t_m) in M, and (3) a proper tree T_{pI} in $G[T_{pmax}(\Sigma)+l_k;\bar{T}_{pmax}(\Sigma)+l_i\}$ whose magnetic graph is $M[l_i;l_k\}$.

From (3), there exists a proper tree represented by $T_{pI}+\Gamma_q$ in $G[T_{pmax}(\Sigma)-\Gamma_q+l_k;\bar{T}_{pmax}(\Sigma)-S_p+l_i\}$ whose magnetic graph is $M[l_i;l_k\}$. Therefore from (1) and (3), there exists a proper tree $T_{pI}+l_k-t_m+S_p+l_i$ in $G[T_{pmax}(\Sigma)-\Gamma_q;\bar{T}_{pmax}(\Sigma)-S_p\}$ whose magnetic graph is M. Let T_{pnew} denote

$$T_{pnew}\stackrel{\Delta}{=}T_{pI}+l_k-t_m+S_p+l_i+T_{pmax}(\Sigma)-\Gamma_q \tag{3-4.15}$$

Then, T_{pnew} is a proper tree in G. Futhermore,

$$|T_{pnew}(C)+\bar{T}_{pnew}(L)|>|T_{pmax}(C)+\bar{T}_{pmax}(L)|, \tag{3-4.16}$$

which contradicts that T_{pmax} is a maximum proper tree. Therefore

$$\mathsf{F}_{St}\widetilde{\mathsf{M}}\mathsf{F}_{l\Gamma}\equiv 0. \tag{3-4.17}$$

In the similar manner, it can be proved that the matrix-products $\mathsf{F}_{St}\widetilde{\mathsf{M}}\mathsf{F}_{lG}$, $\mathsf{F}_{Rt}\widetilde{\mathsf{M}}\mathsf{F}_{l\Gamma}$, $\mathsf{F}'_{lG}\widetilde{\mathsf{M}}'\mathsf{F}'_{St}$, $\mathsf{F}'_{l\Gamma}\widetilde{\mathsf{M}}'\mathsf{F}'_{St}$ and $\mathsf{F}'_{l\Gamma}\widetilde{\mathsf{M}}'\mathsf{F}'_{Rt}$ are all zero-matrices.

Q.E.D.

From Lemma 3-4.1, Eq.(3-4.6) and (3-4.7) become

$$\mathsf{v}_S-\tilde{\mathsf{e}}_S=-\tilde{\mathsf{F}}_{SC}\mathsf{v}_C \quad (a) \tag{3-4.18}$$

$$\mathsf{v}_R-\tilde{\mathsf{e}}_R=-\tilde{\mathsf{F}}_{RC}\mathsf{v}_C-\tilde{\mathsf{F}}_{RG}\mathsf{v}_G \quad (b)$$

$$\mathsf{v}_L-\tilde{\mathsf{e}}_L=-\tilde{\mathsf{F}}_{LC}\mathsf{v}_C-\tilde{\mathsf{F}}_{LG}\mathsf{v}_G-\tilde{\mathsf{F}}_{L\Gamma}\mathsf{v}_\Gamma \quad (c)$$

$$\mathsf{i}_C-\check{\mathsf{j}}_C=\tilde{\mathsf{F}}'_{SC}\mathsf{i}_S+\tilde{\mathsf{F}}'_{RC}\mathsf{i}_R+\hat{\mathsf{F}}'_{LC}\mathsf{i}_L \quad (a) \tag{3-4.19}$$

$$\mathsf{i}_G-\hat{\mathsf{j}}_G=\hat{\mathsf{F}}'_{RG}\mathsf{i}_R+\hat{\mathsf{F}}'_{LG}\mathsf{i}_L \quad (b)$$

$$\mathsf{i}_\Gamma-\hat{\mathsf{j}}_\Gamma=\hat{\mathsf{F}}'_{L\Gamma}\mathsf{i}_L, \quad (c)$$

where

$$\tilde{\mathsf{F}}_{SC}=\mathsf{F}_{SC}-\mathsf{F}_{St}\grave{\mathsf{M}}\mathsf{F}_{lC},\qquad \hat{\mathsf{F}}_{RC}=\mathsf{F}_{RC}-\mathsf{F}_{Rt}\acute{\mathsf{M}}\mathsf{F}_{lC},\qquad \tilde{\mathsf{F}}_{RG}=\mathsf{F}_{RG}-\mathsf{F}_{Rt}\widetilde{\mathsf{M}}\mathsf{F}_{lG}$$

$\tilde{F}_{LC}=F_{LC}-F_{Lt}\tilde{M}F_{1C}$, $\tilde{F}_{LG}=F_{LG}-F_{Lt}\tilde{M}F_{1G}$, $\tilde{F}_{L\Gamma}=F_{L\Gamma}-F_{Lt}\tilde{M}F_{1\Gamma}$

$\tilde{e}_S = e_S-F_{St}\tilde{M}e_1$, $\tilde{e}_R = e_R-F_{Rt}\tilde{M}e_1$, $\tilde{e}_L = e_L-F_{Lt}\tilde{M}e_1$

$\tilde{j}_C = j_C-F'_{1C}\tilde{M}j_t$, $\tilde{j}_G = j_G-F'_{1G}\tilde{M}j_t$, $\tilde{j}_\Gamma = j_\Gamma-F'_{1\Gamma}\tilde{M}j_t$.

Eq.(3-4.18) and (3-4.19) are ones where F.. are replaced with F.. in Eq.(2-3.9) and (2-3.10).

Since the voltage vs current relations of the network-elements are specified by Eq.(2-3.8), a matric form of state equations of an RLCT network may be written as

$$\dot{x}=Ax+Bu, \qquad (3\text{-}4.20)$$

where

$$x=\begin{vmatrix} v_C \\ i_L \end{vmatrix}, \quad A=\begin{vmatrix} \mathcal{C}^{-1}, & 0 \\ 0 & ,\mathcal{L}^{-1} \end{vmatrix}\begin{vmatrix} -\mathcal{Y}, & H \\ -H', & -\mathcal{Z} \end{vmatrix}$$

$$B=\begin{vmatrix} \mathcal{C}^{-1}, & 0 \\ 0 & ,\mathcal{L}^{-1} \end{vmatrix}\begin{vmatrix} 1,\tilde{F}'_{RC}\mathcal{R}^{-1}\tilde{F}_{RG}G_2^{-1}, & 0 & ,\tilde{F}'_{SC}C_1, & \tilde{F}'_{RC}\mathcal{R}^{-1} & ,0 \\ 0, & -\tilde{F}_{LG}\mathcal{g}^{-1} & ,-\tilde{F}_{L\Gamma}L_2, & 0 & ,-\tilde{F}_{L\Gamma}\mathcal{g}^{-1}\tilde{F}'_{RG}R_2^{-1} & ,1 \end{vmatrix}$$

$$u=\begin{vmatrix} \tilde{j}_C \\ \tilde{j}_G \\ \dot{\tilde{j}}_\Gamma \\ \dot{\tilde{e}}_S \\ \tilde{e}_R \\ \tilde{e}_L \end{vmatrix}, \quad \begin{aligned} &\mathcal{C}=C_2+\tilde{F}'_{SC}C_1\tilde{F}_{SC}, \quad \mathcal{L}=L_1+\tilde{F}_{L\Gamma}L_2\tilde{F}'_{L\Gamma} \\ &\mathcal{Y}=\tilde{F}'_{RC}\mathcal{R}^{-1}\tilde{F}_{RC}, \quad \mathcal{Z}=\tilde{F}_{LG}\mathcal{g}^{-1}\tilde{F}'_{LG} \\ &H=\tilde{F}'_{LC}-\tilde{F}_{LG}\mathcal{g}^{-1}\tilde{F}'_{RG}G_1\tilde{F}_{RC} \\ &\mathcal{R}=R_1+\tilde{F}_{RG}G_2^{-1}\tilde{F}'_{RG}, \quad \mathcal{g}=G_2+\tilde{F}'_{RG}R_1^{-1}\tilde{F}_{RG} \end{aligned}$$

In an RLC network, Property 2-3.1 - 2-3.4 hold. In an RLCT network, consider properties of state equations.

Property 3-4.1

The matrices $\mathcal{C}$, $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{g}$ are positive definite unless there are no special relations among the network-element-values.

Proof: Since the matrix $\mathcal{C}$ is rewritten as

$$C=[1,-F'_{SC}]\begin{vmatrix}C_2, & 0\\ 0 & ,C_1\end{vmatrix}\begin{vmatrix}1\\ -F_{SC}\end{vmatrix}, \qquad (3\text{-}4.21)$$

which is positive semi-definite. Since there are no special relations among the network-element-values, $|C|\neq 0$. Then, C is positive definite. Similarly the matrix L is positive definite. Similarly the matrices R and g can be proved to be positive definite.

Q.E.D.

A property similar to Property 2-3.2 does not hold since there may exist elements of $\widetilde{F}_{LC}$ whose absolute values are greater than 1.

Property 3-4.2

If the state equations contain derivatives of input functions, there exists at least one set of branches which consists of capacitor-branches"inductor-branches" and voltage-source-branches "current-source-branches" only, which is the ring-sum of some loops"cut-sets" in B-S-graphs[@] of G^* {all resistor-, inductor- and current-source-branches} and M"S-B-graphs[@] of G^* [all capacitor-, resistor- and voltage-source-branches] and M".

Proof: See Appendix IV .

Q.E.D.

Property 3-4.3

If there exists at least one set of branches containing capacitor-branches"inductor-branches" which is the ring-sum of

@ See Appendix IV .

some cut-sets"loops" in S-B-graphs of G[all resisitor- and inductor-branches] and M"B-S-graphs of G{all capacitor- and resistor-branches} and M", the matrix A of the state equation is singular.

Proof: See Appendix IV.

Q.E.D.

3-5 CONCLUDING REMARKS

Based on proper trees, the problems on the solvability-conditions and the formulation of state equations are discussed in the previous sections. The networks without proper trees are ones which do not satisfy the Kirchhoff's laws, or which are physically meanless. Consider the matrices,

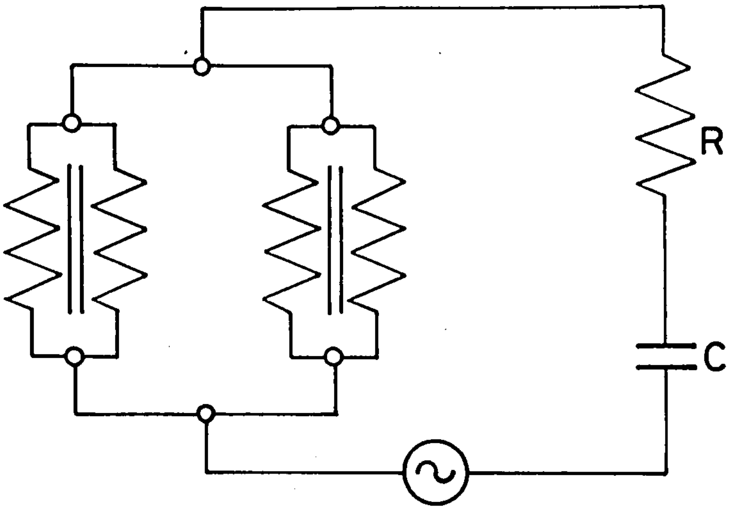
$$Q_A \triangleq \begin{vmatrix} Q_\Sigma , Q_M \\ 0 \ , B_W \end{vmatrix} \qquad B_A \triangleq \begin{vmatrix} B_\Sigma , B_M \\ 0 \ , Q_W \end{vmatrix}$$

(noting that $B_W Q_W' = 0$.)

The row-rank of Q_A"B_A", derived from networks without proper trees, is not equal to the number of rows. Then there are cut-sets"loop" containing only transformer-branches in the electric graphs. A simple example is shown in Fig.3-5.1(a), where the network has no proper tree. It has no unique solution.

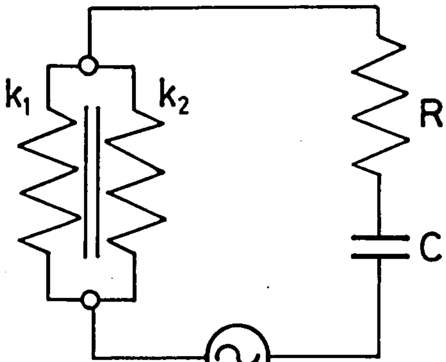
The network shown in Fig.3-5.1(b) has proper trees, however, if $k_1 = k_2$, it has no unique solution.

Consider whether there exists a network which is equivalent to a given network N and contains less ideal-transformers than N. If there exists a graph (denoted by G_A) whose basic cut-set matrix is Q_A, then the network corresponding to G_A may have



(a)

Fig. 3-5.1
(a) Nėtwork without proper trees.
(b) Network without unique solution if $k_1=k_2$.



(b)

no ideal transformers and is equivalent to N. If there exists a graph whose basic cut-set matrix is a proper submatrix of Q_A, and cannot be Q_A, then the remaining submatrix of Q_A corresponds to ideal transformers which is contained in a network which is equivalent to N and contains less ideal-transformers than N.

CHAPTER 4 RCG NETWORK(2)(3)

4-1 INTRODUCTION

The network considered in this chapter contains resistors, capacitors, two-port-ideal gyrators and independent sources. Furthermore it is assumed to satisfy Assumption A in Chapter 2.

Several papers[6][12][13] on the solvability have been published. E.S.Kuh and *et.al.*[6] gave two basic conjectures on the solvability and the order of complexity. K.Abdullah and *et. al.*[12] proved that general passive networks are terminal solvable[@], but did not touch completely solvability. M.M.Milic[13][14] has proved the conjectures given by [6] using the thorem that the determinant of an even order skew-symmetric is a 2nd order polynominal in its elements.

Several papers[6][15] on the formulations of explicit form of state equations, have been published. They formulate the state equations of the networks under certain special assumptions.

In Section 4-2, the properties of multi-port-ideal gyrators are studied. It is pointed out that instead of a linear passive network, its equivalent RCG network may be analized

In Section 4-3, conditions on the solvability and the order of complexity are studied.

In Section 4-4, an explicit form of state equations based on the solvability condition obtained in Section 4-3 is formulated.

@ That is, the complementary variables of the sources associated with two-terminal elements can be determined uniquely but the other variables may not be determined uniquely.

4-2 MULTI-PORT GYRATOR

A multi-port gyrator is a passive element. Let v_g and i_g denote port-voltage- and current-vectors, respectively, of a multi-port gyrator, whose port-characteristic is given by

$$v_g=R_G i_g \qquad \text{or } i_g=G_G v_g, \tag{4-2.1}$$

where $R_G'=-R_G$ or $G_G'=-G_G$.

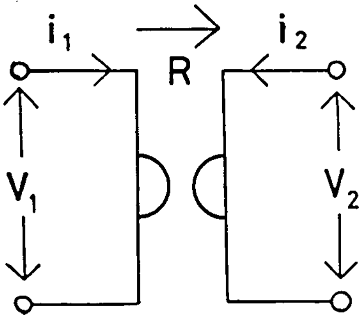
It is well-known that for a skew-symmetric matrix R_G, there exists a nonsingular matrix P such that[16]

$$R_G=P(E\dotplus 0)P', \tag{4-2.2}$$

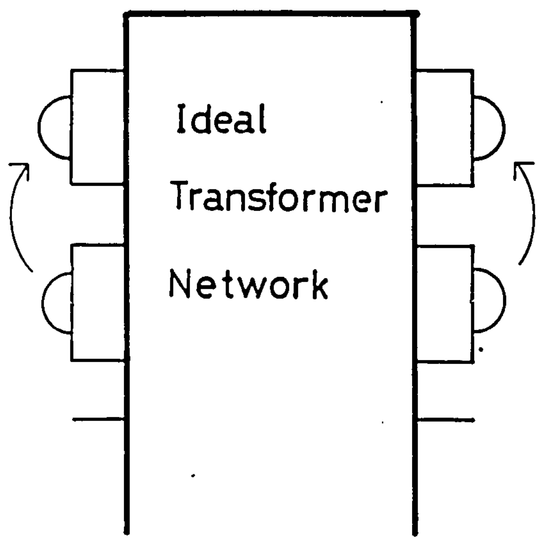
where $E\triangleq E_1\dotplus E_1\dotplus\cdots\dotplus E_1$, $\quad E_1\triangleq\begin{vmatrix}0,1\\-1,0\end{vmatrix}$. $\qquad$ (4-2.3)

The matrix E_1 represents a port-characteristic of a two-port gyrator, the symbol of which is shown in Fig.4-2.1(a).

As mentioned in Section 3-2, any multiwinding transformer whose port charateristic is represent is specified by a symmetric



(a)



(b)

Fig.4-2.1 (a) Symbol of two-prot gyrator. (b) Equivalent network of multi-port gyrator.

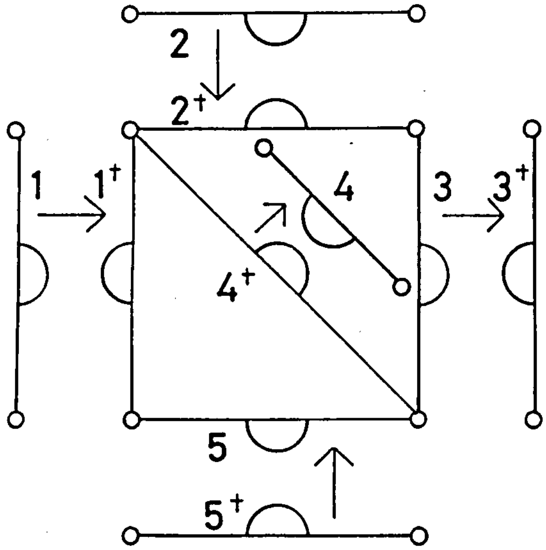


Fig.4-2.2 Equivalent network of ideal transformer.

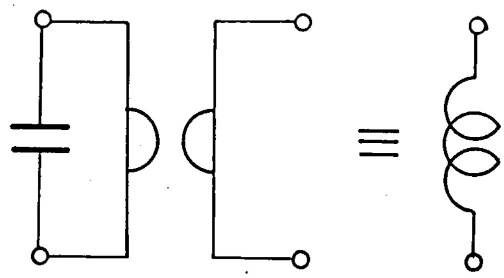


Fig.4-2.3 Equivalent network of inductor.

positive semi-definite matrix, is equivalent to a network which consists only of ideal transformers and inductors whithout mutual coupling. It can be seen from Eq.(4-2.2) that any multi-port gyrator is equivalent to a network which consists only of ideal transformers and two-port gyrators(Fig.4-2.1(b)). Furthermore, any ideal transformer is equivalent to a network which consists only of two-port gyrators. The ideal transformer in Fig.3-2.2 is equivalent to the network shown in Fig.4-2.2. It is well known that any inductor is equivalent to a cascade connection of a two-port gyrator and a capacitor as shown in Fig.4-2.3. Consequently, a linear passive nonreciprocal network can be converted to an RCG network, and the RCG network obtained may be analized instead of the linear passive nonreciprocal network.

The voltage-current-relation of two-port gyrators in a network is given by

$$v_G = G i_G, \qquad (4\text{-}2.4)$$

where the matrix given by

$$G = E G_D, \qquad (4\text{-}2.5)$$

where

$$G_D = G_1 \dotplus \cdots \dotplus G_{n_G}$$

$$G_i = \begin{vmatrix} r_i, & 0 \\ 0 & , r_i \end{vmatrix}$$

n_G: number of two-port gyrators.

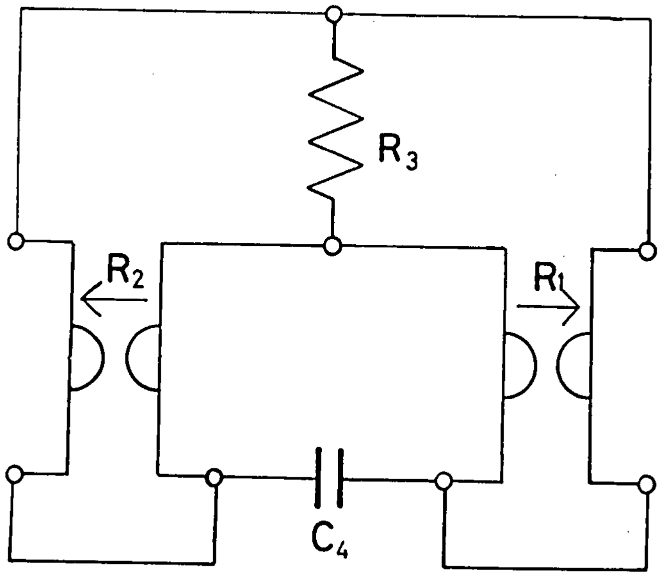
(Note that the subscript G denotes two-port gyrators and matrix G denotes the resistance matrix of two-port gyrators, although in Chapter 2 and 3, the subscpipt G denotes twig-resistors and the matrix G denotes the conductance matrix of twig-resistors.)

AN RCG NETWORK AND ITS GRAPH

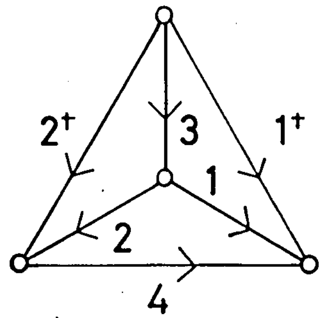
A graph for a two-port gyrator is obtained by replacing each port of the gyrator with a branch which is oriented from a node j to j'. A graph G corresponding to an RCG network N is obtained as shown in Fig.4-2.4. If one branch corresponding to a port of gyrator is labeled with i, the other is assumed to be labeled with i^+. Two branches i and i^+ for a two-port gyrator are called a pair of gyrator-branches.

Definition[6] 4-2.1

A proper tree T_p in an RCG network is a tree of the graph G where every pair of gyrator branches are either in the tree or in its cotree.



(a)



(b)

Fig. 4-2.4 (a) RCG network and (b) its graph.

Definition[6] 4-2.2

A maximum proper tree T_{pmax} for an RCG network is defined to be a proper tree having maximum $|T_p(C)|$.

Let us introduce another graph denoted by G^+ induced from the graph G for an RCG network.

Definition 4-2.3

A graph G^+ induced from the graph G for an RCG network is defined as

1) Two graphs G and G^+ except labelings and orientations are isomorphic.

2) For non-gyrator-branches, the labeling and the orientations of every branch in G^+ is the same as that in G.

3) For gyrator branches, a gyrator-branch in G^+ corresponding to

a gyrator-branch i in G is labeled with i^+, and it is oriented in the same direction as the branch i in G. A gyrator-branch in G^+ corresponding to a gyrator branch i^+ in G is labeled with i, and it is oriented in the opposite direction as the branch i^+ and G.

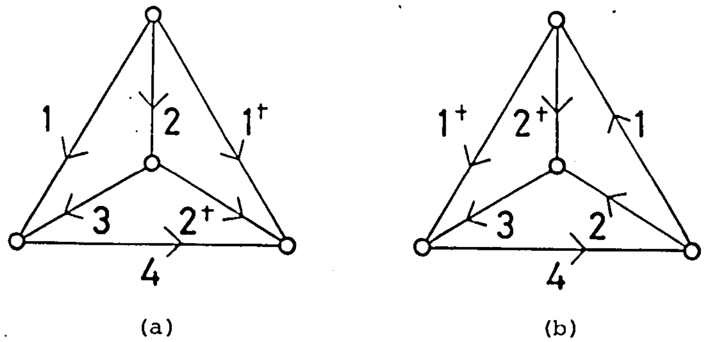


Fig.4-2.5 Example for (a) G and (b) G^+.

The graph G and G^+ for the network shown in Fig.4-2.4(a) are shown in Fig.4-2.5 (a) and (b), respectively.

Definition 4-2.4

A common tree T_c of G and G^+ is a tree in both G and G^+.

The mark $^+$ may be regarded as an operator for a branch i or a set of branches, that is,

i) for gyrator branches, $(i)^+=i^+$ $(i^+)^+=i$

ii) for non-gyrator branches, $(i)^+=i$

iii) for a set of branches S, S^+ is a set of branches which contains exactly every $(i)^+$ for branch i in S.

For a common tree T_c, define a set S_g as

$$S_g=T_c \cup T_c^+ - T_c \cap T_c^+ \qquad (4-2.6)$$

The set S_g consists only of gyrator branches, and if a branch is in S_g, its pair-branch is in S_g.

Define a set S_{gm} which includes either of every pair of gyrator branches in S_g, then

$$S_g=S_{gm}+S_{gm}^+ \qquad S_{gm}\cap S_{gm}^+=\phi \tag{4-2.7}$$

Definition 4-2.5

For a common tree T_c, there may exist a common tree $T_{cm}=(T_c\cap T_c^+)\cup S_{gm}$, which is called a common tree of the same class of T_c.

Lemma 4-2.1

If T_c is a common tree of G and G^+, then T_c^+ is a common tree of G and G^+.

Proof: Every branch i in G corresponds to a branch $(i)^+$ in G^+, and every branch i in G^+ corresponds to a branch $(i)^+$ in G. From Definition 4-2.3, if T is a tree of G, then T^+ is a tree of G^+, and *vice versa*. If T_c is a common tree of G and G^+, T_c^+ is a common tree of G^+ and G.

Q.E.D.

From Definition 4-2.2 and 4-2.3, we obtain

Lemma 4-2.2

A proper tree T_p for an RCG network is a common tree of its graphes G and G^+.

Proof: Since T_p is a tree of G, $(T_p)^+$ is a tree of G^+. Since $(T_p)^+=T_p$, T_p is a common tree of G and G^+.

Q.E.D.

4-3 NETWORK EQUATION AND UNIQUE SOLVABILITY

In this section, unique solvability of an RCG network is studied. Solvability of the network may depend not only on the network-topology but also on the network-element's values.

In Section 4-3.1, solvability-conditions depending only on the network-topology (so called topological condition on solvability) are studied.

In Section 4-3.2, solvability-conditions depending on the network-element's values are studied. (overall solvability)

4-3.1 TOPOLOGICAL CONDITIONS ON SOLVABILITY

Network equations of an RCG network containing n_C capacitors, n_R resistors and n_G two-port gyrators are given as

$$\boldsymbol{\theta}(p)\begin{vmatrix} \mathrm{i} \\ \mathrm{v} \end{vmatrix} \triangleq \begin{array}{c} \begin{array}{cccccc} (n_C) & (n_R) & (2n_G) & (n_C) & (n_R) & (2n_G) \end{array} \\ \begin{vmatrix} Q_C, & Q_R, & Q_G, & 0, & 0, & 0 \\ 0, & 0, & 0, & B_C, & B_R, & B_G \\ -1, & 0, & 0, & Cp, & 0, & 0 \\ 0, & R, & 0, & 0, & -1, & 0 \\ 0, & 0, & G, & 0, & 0, & -1 \end{vmatrix} \end{array} \begin{vmatrix} \mathrm{i}_C \\ \mathrm{i}_R \\ \mathrm{i}_G \\ \mathrm{v}_C \\ \mathrm{v}_R \\ \mathrm{v}_G \end{vmatrix} = \begin{vmatrix} \mathrm{j} \\ \mathrm{e} \\ 0 \\ 0 \\ 0 \end{vmatrix}, \qquad (4\text{-}3.1)$$

where the subscript G denotes gyrators. The matrix $Q \triangleq [Q_C, Q_R, Q_G]$ "$B \triangleq [B_C, B_R, B_G]$" is the basic cut-set"the basic loop" matrix

A necessary and sufficient condition that the RCG network has a unique solution is that the coefficient matrix $\boldsymbol{\theta}(p)$ is non-singular.

Let us consider what properties the network-topology has if $\boldsymbol{\theta}(p)$ is nonsingular. The matrix $\boldsymbol{\theta}(p)$ is transformed into

$$\left|\begin{array}{ccc|ccc} Q_C, & Q_R, & Q_G & 0, & 0, & 0 \\ B_C C^{-1} p^{-1}, & B_R R, & B_G G & B_C, & B_R, & B_G \\ \hline & & & C, & 0, & 0 \\ & 0 & & 0, & -1, & 0 \\ & & & 0, & 0, & -1 \end{array}\right| \qquad (4\text{-}3.2)$$

or

$$\left|\begin{array}{ccc|ccc} Q_C, & Q_R, & Q_G & Q_C C, & Q_R R^{-1}, & Q_G G^{-1} \\ 0, & 0, & 0 & B_C, & B_R, & B_G \\ \hline -1, & 0, & 0 & & & \\ 0, & R, & 0 & & 0 & \\ 0, & 0, & G & & & \end{array}\right|, \qquad (4\text{-}3.3)$$

by multiplying the matrices,

$$\left|\begin{array}{ccc|ccc} 1, & 0, & 0 & & & \\ 0, & 1, & 0 & & 0 & \\ 0, & 0, & 1 & & & \\ \hline C^{-1}p^{-1}, & 0, & 0 & 1, & 0, & 0 \\ 0, & R, & 0 & 0, & 1, & 0 \\ 0, & 0, & 0 & 0, & 0, & 1 \end{array}\right| \quad \text{or} \quad \left|\begin{array}{ccc|ccc} 1, & 0, & 0 & C, & 0, & 0 \\ 0, & 1, & 0 & 0, & R^{-1}, & 0 \\ 0, & 0, & 1 & 0, & 0, & G^{-1} \\ \hline & & & 1, & 0, & 0 \\ & 0 & & 0, & 1, & 0 \\ & & & 0, & 0, & 1 \end{array}\right|$$

from the right, respectively. Here the dotted lines show the partitioning of the matrices.

Since the lower right partitioned submatrix of (4-3.2) is a nonsingular diagonal matrix, the upper left submatrix(denoted by A_1) is nonsingular if $\Theta(p)$ is nonsingular and *vice versa*. Similarly the upper right submatrix of (4-3.3)(denoted by A_2) is nonsigular if $\Theta(p)$ is nonsingular and *vice versa*. Since the matrices A_1 and A_2 are of the dimension $n_C+n_R+2n_G$, the product of $A_1 \cdot A_2'$ exists and

$$A_1A_2' = \begin{vmatrix} A_{11}, A_{12} \\ A_{21}, A_{22} \end{vmatrix}, \qquad (4\text{-}3.4)$$

where

$$A_{11} = Q_C C p Q_C' + Q_R R^{-1} Q_R' + Q_G (G^{-1})' Q_G'$$

$$A_{12} = Q_C B_C' + Q_R B_R' + Q_G B_G'$$

$$A_{21} = B_C Q_C' + B_R Q_R' + B_G G (G^{-1})' Q_G'$$

$$A_{22} = B_C C^{-1} p^{-1} B_C' + B_R R B_R' + B_G G B_G'$$

The matrix A_{11}"A_{22}" is known as the cut-set-admittance"the loop-impedance" matrix. The matrix A_{12} is a zero matrix because of the orthogonality of the cut-set- and the loop matrices of a graph. Therefore, the matrices A_{11} and A_{22} are nonsingular if the matrix A_1A_2' is nonsingular and *vice versa*.

Using the matrix G_D defined in (4-2.5), the matrices A_{11} and A_{22} may be rewritten as

$$A_{11} = [Q_C, Q_R, Q_G E] \begin{vmatrix} Cp, & 0, & 0 \\ 0, & R^{-1}, & 0 \\ 0, & 0, & G_D^{-1} \end{vmatrix} \begin{vmatrix} Q_C' \\ Q_R' \\ Q_G' \end{vmatrix} \qquad (4\text{-}3.5)$$

$$A_{22} = [B_C, B_R, B_G E] \begin{vmatrix} C^{-1}p^{-1}, & 0, & 0 \\ 0, & R, & 0 \\ 0, & 0, & G_D \end{vmatrix} \begin{vmatrix} B_C' \\ B_R' \\ B_G' \end{vmatrix}. \qquad (4\text{-}3.6)$$

The last matrix on the right hand-side of Eq.(4-3.5)"Eq.(4-3.6)" is the transposed matrix of the basic cut-set"loop" matrix. The second matrix on the right hand-side of Eq.(4-3.5)"Eq.(4-3.6)" is a diagonal matrix.

The first matrix on the right hand-side of Eq.(4-3.5)"Eq.(4-3.6)" is the matrix whose submatrix corresponding to the non-gyrator elements is identical to the submatrix of the basic cut-set"loop" matrix corresponding to non-gyrator-elements, and whose submatrix corresponding to the gyrators is the matrix obtained from the submatrix of the basic cut-set"loop" matrix corresponding to the gyrators by multipling the matrix E from the right.

The multiplication of the matrix E from the right means that every pair of columns corresponding to every pair of gyrator-branches are interchanged and every one column of the pair is multipled by -1. From a topological point of view, the first matrix on the right hand-side of Eq.(4-3.5)"Eq.(4-3.6)" is a basic cut-set"loop" matrix of the graph G^{+} defined in Definition 4-2.3. Then the matrices A_{11} and A_{22} may be written as

$$A_{11}=Q_{+}Y_{D}Q' \tag{4-3.7}$$

$$A_{22}=B_{+}Z_{D}B', \tag{4-3.8}$$

where Q_{+}"B_{+}" is a basic cut-set"loop" matrix of G^{+}. The matrix Y_D and Z_D are, respectively,

$$Y_D=\begin{vmatrix} Cp, & 0 & , 0 \\ 0 & ,R^{-1}, & 0 \\ 0 & , 0 & ,G_D^{-1} \end{vmatrix} \quad \text{and} \quad Z_D=\begin{vmatrix} C^{-1}p^{-1} & ,0, & 0 \\ 0 & ,R, & 0 \\ 0 & ,0, & G_D \end{vmatrix} \tag{4-3.9}$$

The columns for a nonzero majar determinant of Q_{+}"B_{+}" correspond to a tree"cotree" of G^{+}. The columns for a nonzero majar determinant of Q"B" correspond to a tree"cotree" of G. There are some relations between the determinant $|A_{11}|$"$|A_{22}|$" and trees "cotree" in G and G^{+}.

Property 4-3.1

By Binet-Cauchy theorem, the determinant $|A_{11}|$"$|A_{22}|$" may be written as

$$|A_{11}|\text{"}|A_{22}|\text{"}=\Sigma\{\textit{Products of corresponding majar determinants of } Q_{+}Y_{D}\text{"}B_{+}Z_{D}\text{" and } Q'\text{"}B'\text{"}.\}, \tag{4-3.10}$$

where the summation is over all such majar determinants.

A nonzero product of the majar determinants of Q_+Y_D"B_+Z_D" and Q'"B'" corresponds to a tree"a cotree" of G^+ and a tree"a cotree" of G, that is, to a common tree"a common cotree" of G and G^+.

Note that the products of the majar determinants are all positive in case of an RCL network, but they may not be all positive in Eq.(4-3.10).

From Definition 4-2.5 and Property 4-3.1, we obtain

Property 4-3.2

Let α"β" be the product of majar determinants corresponding to a common tree T_c"T_{cm}, a common tree of the same class of T_c".

$$\begin{aligned}&\text{If } |K| \text{ is even, } \alpha=\beta,\\ &\text{If } |K| \text{ is odd, } \alpha=-\beta,\end{aligned} \tag{4-3.11}$$

where $K \triangleq T_c - T_c \cap T_{cm}$.

Proof: From Lemma 4-2.1, T_c^+ and T_{cm}^+ common trees of G and G^+. Since T_{cm} is a common tree of the same class of T_c, $T_c \cap T_c^+ = T_{cm} \cap T_{cm}^+$. Let U and W denote $T_c \cap T_c^+$ and $T_c - T_c \cap T_{cm}^+$, respectively, then the following equalities hold;

$U \cap W = \phi$, $U \cap K = \phi$, $W \cap K = \phi$.

Then we obtain

$T_c = U+W+K$, $T_c^+ = U+W^+ + K^+$, $T_{cm} = U+W+K^+$, $T_{cm}^+ = U+W^+ + K$.

Let γ_W"γ_W^+", γ_K"γ_K^+" and γ_U denote the majar determinants for W"W^+", K"K^+" and U of the submatrices of Q corresponding to $G[U,K]$, $G[U,W]$ and $G[K,W]$, respectively. Let γ_{QT_c}"$\gamma_{QT_{cm}}$" and $\gamma_{Q_+T_c}$"$\gamma_{Q_+T_{cm}}$" denote the majar determinants of Q and Q_+ for T_c"T_{cm}", respectively. Then the following equalities hold;

$\gamma_{QT_c}=\gamma_U\gamma_W\gamma_K$, $\quad \gamma_{QT_{cm}}=\gamma_U\gamma_W\gamma_K^+$.

Let n_W"n_K" denote the number of the elements with the mark '+' in W"K", and $m_K\triangleq|K|-n_K$. Then we obtain

$$\gamma_{Q_+T_c}=\gamma_U(-1)^{n_W}\gamma_W(-1)^{n_K}\gamma_K^+, \qquad \gamma_{Q_+T_{cm}}=\gamma_U(-1)^{n_W}\gamma_W^+(-1)^{m_K}\gamma_K.$$

Therefore we obtain

$$\alpha=(-1)^{n_W}(-1)^{n_K}\gamma_U^2\gamma_W\gamma_W^+\gamma_K\gamma_K^+\ Y, \qquad \beta=(-1)^{n_W}(-1)^{m_K}\gamma_U^2\gamma_W\gamma_W^+\gamma_K\gamma_K^+\ Y,$$

where Y is the admittance-product for T_c"or T_{cm}".

Then if $|K|$ is even, $\alpha=\beta$, and if $|K|$ is odd, $\alpha=-\beta$.

Q.E.D.

Property 4-3.1 states that a necessary condition[15] that an RCG network has a unique solution is that there exists a common tree in G and G^+. Property 4-3.2 states that even if there exists a common tree in G and G^+, the network may not have a unique solution. Since even if there exists a common tree of the same class of T_c, where $|K|$ is even, there may exist another common tree of the same class of T_c, where $|K|$ is odd, we introduce a special common tree of G and G^+.

Definition 4-3.1

A common tree"cotree" corresponding to a product of majar determinants of Q_+Y_D"B_+Z_D" and Q'"B'" which remains in the expansion of the determinant $|A_{11}|$"$|A_{22}|$" without cancellation is called an independent common tree"cotree".

From the above definition, if $|A_{11}|\neq 0$"$|A_{22}|\neq 0$", there exists at least one independent common tree"cotree" in G and G^+.

Lemma 4-3.1

If there exists an independent common tree in G and G^+, there exists at least one proper tree in G.

Proof: Let T_c be an independent common tree in G and G^+. If $S_g=\phi$,@ T_c is a proper tree. If $S_g \neq \phi$, consider graphes $G[T_c \cap T_c^+; \bar{T}_c \cap \bar{T}_c^+]$ and $G^+[T_c \cap T_c^+; \bar{T}_c \cap \bar{T}_c^+]$ (denoted by G_g and G_g^+, respectively). The basic cut-set matrix of G_g and G_g^+ as denoted by Q_g and Q_{+g} exist such that

$$Q_{+g}=Q_g E, \tag{4-3.12}$$

where the dimension of E is equal to $|S_g|$. The submatrix of A_{11} for S_g is $Q_g E G_{Dg}^{-1} Q_g'$, where G_{Dg} is the submatrix of G_D for S_g. This matrix-product is a skew-symmeric matrix. Its determinant is written as

$$|Q_g E G_{Dg}^{-1} Q_g'| = \{p(\cdots, x_{ij}, \cdots)\}^2, \tag{4-3.12}$$

where $p(\cdots, x_{ij}, \cdots)$ is a pfaffian[16] and x_{ij} is an (i,j) element of the matrix $Q_g E G_{Dg}^{-1} Q_g'$. The element x_{ij} is a sum of r_k^{-1}, where r_k is defined in Eq.(4-3.5). Therefore there exists one term represented by $(r_{k1}^{-1}, r_{k2}^{-1}, \cdots, r_{ki}^{-1}, \cdots)^2$ in the determinant $|Q_g E G_{Dg}^{-1} Q_g'|$, which corresponds to a proper tree in G_g. Consequently, there exists a proper tree in G if there exists an independent common tree in G and G^+.

Similarly it can be proved that there exists a proper tree if there exists an independent common cotree in G and G^+.

Q.E.D.

Fom the proof of Lemma 4-3.1, we obtain

@ See Eq.(4-2.6)

Corollary

There exists a proper tree which contains any one branch of an independent common tree. And there exists a proper tree which contains $T_c \cap T_c^+$, where T_c is an independent common tree.

From Lemma 4-3.1, we obtain

Theorem 4-3.1

A necessary condition for an RCG network to have a unique solution is that there exists at least one proper tree.

we obtain the theorem on the order of complexity as

Theorem 4-3.2

The order of complexity of an RCG network is not greater than $|T_{pmax}(C)|$, where T_{pmax} is a maximum proper tree.

Proof: Let D_1 and D_2 denote the right lower submatrix of (4-3.2) and the left lower submatrix of (4-3.3), respectively, that is,

$$D_1 = \begin{vmatrix} Cp, & 0, & 0 \\ 0, & -1, & 0 \\ 0, & 0, & -1 \end{vmatrix}, \quad (4\text{-}3.13) \qquad D_2 = \begin{vmatrix} -1, & 0, & 0 \\ 0, & R, & 0 \\ 0, & 0, & G \end{vmatrix}. \quad (4\text{-}3.14)$$

We obtain

$$|\Theta(p)|^2 = |D_1||D_2||A_{11}||A_{22}|. \qquad (4\text{-}3.15)$$

Since the maximum degree of p in $|A_{11}|$ is $|T_{pmax}(C)|$ and the minimum degree of p^{-1} in $|A_{22}|$ is $n_C - T_{pmax}(C)$, the maximum degree in $|A_{11}||A_{22}|$ is $2|T_{pmax}(C)| - n_C$. From Eq.(4-3.15), the maximum degree of p in $|\Theta(p)|^2$ is $2|T_{pmax}(C)|$. Consequently, the maximum degree of p in $|\Theta(p)|$ is $|T_{pmax}(C)|$.

Q.E.D.

4-3-2 SOLVABILITY DEPENDING UPON NETWORK-ELEMENT-VALUES

We have studied a topological solvability-condition in Section 4-3-1. If an RCG network has a unique solution, there exists at least one proper tree, but the converse is not always true. It depends on network-element-values. In this section, it is studied how network-element-values affect solvability.

Assume that there exists at least one proper tree in the graph corresponding to a given RCG network. A network equation of the network is written as

$$\theta(p)\begin{vmatrix} i \\ v \end{vmatrix} = \begin{vmatrix} j \\ e \\ 0 \end{vmatrix}, \tag{4-3.16}$$

where

$$\theta(p)=\begin{vmatrix} -F',1 & 0 \\ 0 & 1,F \\ Z & Y \end{vmatrix}, \qquad F=\begin{vmatrix} F_{CC}, F_{CR}, F_{CG} \\ F_{RC}, F_{RR}, F_{RG} \\ F_{GC}, F_{GR}, F_{GG} \end{vmatrix},$$

$$Z=\begin{vmatrix} 1, & & & & & \\ & R_1, & & & & \\ & & 1, & & & \\ & & & R_2, & & \\ & & & & G_1, & \\ & & & & & G_2 \end{vmatrix}, \qquad Y=\begin{vmatrix} -C_1p, & & & & & \\ & -1, & & & & \\ & & -C_2p, & & & \\ & & & -1, & & \\ & & & & -1, & \\ & & & & & -1 \end{vmatrix},$$

$$i=\begin{vmatrix} i_1 \\ i_2 \end{vmatrix}, \quad i_1=\begin{vmatrix} i_{C1} \\ i_{R1} \\ i_{G1} \end{vmatrix}, \quad i_2=\begin{vmatrix} i_{C2} \\ i_{R2} \\ i_{G2} \end{vmatrix},$$

$$v=\begin{vmatrix} v_1 \\ v_2 \end{vmatrix}, \quad v_1=\begin{vmatrix} v_{C1} \\ v_{R1} \\ v_{G1} \end{vmatrix}, \quad v_2=\begin{vmatrix} v_{C2} \\ v_{R2} \\ v_{G2} \end{vmatrix} \tag{4-3.17}$$

The subscripts *1* and *2* represent links and twigs, respectively.

Lemma 4-3.2

If an RCG network has at least one proper tree, the determinant of $\Theta(p)$ is written as

$$|\Theta(p)| = a\prod_{n_G} r_i^2 \sum_{T_p(C,R)} \{\prod_{T_p(C)} C_i p \prod_{\bar{T}_p(R)} R_i (P_{T_p(C,R)})^2\}, \qquad (4\text{-}3.18)$$

where $(P_{T_p(C,R)})^2 = det|G_2^{-1} + F'_{GG} G_1^{-1} F_{GG}|$. The matrix F_{GG} is the fundamental loop matrix in $G[T_p(C,R); \bar{T}_p(C,R)]$. The value $P_{T_p(C,R)}$ is a sum of tree-gyrator admittance products corresponding to independent common trees which contain $T_p(C,R)$. *a=1 or -1.*

Proof: By multipling by the Gauss matrix,

$$\left|\begin{array}{c|c} 1 & \begin{vmatrix} -F', & 1 \\ 0, & 0 \end{vmatrix} Z^{-1} \\ \hline 0 & 1 \end{array}\right|,$$

on the left, the matrix $\Theta(p)$ is transformed into

$$\left|\begin{array}{c|c|c} 0 & -F' \begin{vmatrix} C_1 p, & 0 & 0 \\ 0 & R_1^{-1}, & 0 \\ 0 & 0 & G_1^{-1} \end{vmatrix} & \begin{vmatrix} C_2 p, & & 0 \\ & R_2^{-1}, & \\ 0 & & G_2^{-1} \end{vmatrix} \\ \hline 0 & 1 & F \\ \hline \begin{matrix} 1, & & & & & \\ & R_1, & & 0 & & \\ & & 1, & & & \\ & & & R_2, & & \\ 0 & & & & G_1, & \\ & & & & & G_2 \end{matrix} & \begin{matrix} -C_2 p, & & 0 \\ & -1, & \\ & & \\ & & -1 \\ 0 & & \\ & & \end{matrix} & \begin{matrix} & 0 \\ & \\ C_2 p, & \\ & -1, \\ 0 & \\ & -1 \end{matrix} \end{array}\right| \qquad (4\text{-}3.19)$$

Then, by the Laplace expansion, the determinant of $\Theta(p)$ is represented by the products of the determinants of the left lower submatrices and the right upper submatrices of (4-3.19) as

$$det|\Theta(p)|=a\prod_{n_R} R_j \prod_{n_G} r_i^2 det|QDQ_+^{\prime}|, \qquad (4\text{-}3.20)$$

where

$$Q=\begin{vmatrix} 1,0,0,-F'_{CC},-F'_{RC},-F'_{GC} \\ 0,1,0,-F'_{CR},-F'_{RR},-F'_{GR} \\ 0,0,1,-F'_{CG},-F'_{RG},-F'_{GG} \end{vmatrix}$$

$$D=Diag.\{C_2 p, R_2^{-1}, G_{D2}^{-1}, C_1 p, R_1^{-1}, G_{D1}^{-1}\}$$

$$Q_+=\begin{vmatrix} 1,0,0,-F'_{CC},-F'_{RC},-F'_{GC}E \\ 0,1,0,-F'_{CR},-F'_{RR},-F'_{GR}E \\ 0,0,1,-F'_{CG},-F'_{RG},-F'_{GG}E \end{vmatrix} \qquad (4\text{-}3.21)$$

The dimension of E is $2|T_p(G)|$. The determinant $|QDQ_+^{\prime}|$ is given by a sum of independent common tree-admittance products, therefore $det|\Theta(p)|$ is written as

$$det|\Theta(p)|=a\sum_{T_{cI}} sgn(T_{cI})\prod_{T_{cI}(C)} C_i p \prod_{\bar{T}_{cI}(R)} R_i \prod_{\bar{T}_{cI}(G)} r_i, \qquad (4\text{-}3.22)$$

where T_{cI} is an independent common tree.

$$sgn(T_{cI}) \triangleq |Q|_{T_{cI}} \, |Q_+^{\prime}|_{T_{cI}}$$

From Corollary of Lemma 4-3.1, there exists a proper tree T_p which contains $T_c(C,R)$. Let T_g denote an independent common tree of $G[T_p(C,R);\bar{T}_p(C,R)\}$ and $G^+[T_p(C,R);\bar{T}_p(C,R)\}$. Then Eq.(4-3.22) is rewritten as

$$det|\Theta(p)|=a\sum_{T_p(C,R)} sgn(T_p(C,R))\prod_{T_p(C)} C_i p \prod_{\bar{T}_p(R)} R_i \cdot f(T_p), \qquad (4\text{-}3.23)$$

where $f(T_p)=\sum_{T_g} sgn(T_g)\prod_{\bar{T}_g} r_i$.

Since $f(T_p)$ is represented by the sum of independent common cotree impedance products, then

$$f(T_p)=sgn(T_p(G))\prod_{n_G} r_i^2 |Q_{T_p(G)} G^{-1} Q_{+T_p(G)}|$$

$$\triangleq sgn(T_p(G))\prod_{n_G} r_i^2\, P^2_{T_p(C,R)},$$

where $P_{T_p(C,R)}$ is the pfaffian of the matrix $G_2^{-1}+F'_{GG}G_1^{-1}F_{GG}$. The matrix $Q_{T_p(G)}$ and $Q_{+T_p(G)}$ are basic cut-set matrices of the graphs $G[T_p(C,R);\bar{T}_p(C,R)\}$ and $G^{+}[T_p(C,R);\bar{T}_p(C,R)\}$, respectively. Then Eq.(4-3.23) is rewritten as

$$det|\boldsymbol{\theta}(p)|=a\sum_{T_p(C,R)} sgn(T_p)\prod_{T_p(C)} C_i p \prod_{\bar{T}_p(R)} R_i \prod_{n_G} r_i^2\, P_{T_p(C,R)}, \tag{4-3.24}$$

where $sgn(T_p)=|Q|_{T_p}|Q_+|_{T_p}$.

Let us prove that $sgn(T_p)=1$. From Eq.(4-3.21), we obtain

$$|Q_+|_{T_p}=det\begin{vmatrix}1, 0\\0, E\end{vmatrix}|Q|_{T_p} \tag{4.3.25}$$

Then, $sgn(T_p)=|Q|_{T_p}|Q_+|_{T_p}$

$$=1. \tag{4-3.26}$$

Consequently, we obtain

$$det|\boldsymbol{\theta}(p)|=a\prod_{n_G} r_i^2\sum_{T_p(C,R)}\{\prod_{T_p(C)} C_i p\prod_{\bar{T}_p(R)} R_i (P_{T_p(C,R)})^2\}.$$

Q.E.D.

From Lemma 4-3.2, we obtain

Theorem 4-3.3

A necessary and sufficient condition that an RCG network has a unique solution is that there exists at least one proper tree such that $P^2_{T_p(C,R)}=|G_2^{-1}+F'_{GG}G_1^{-1}F_{GG}|\neq 0$.

We have a sufficient condition depending only on network topology, that is,

Theorem 4-3.4

A sufficient condition that an RCG network has a unique solution is that there exists at least one proper tree T_p such that there exists no other proper tree which contains $T_p(C,R)$.

Proof: Since there exists no other proper tree which contains $T_p(C,R)$, $P_{T_p(C,R)}=|G_2^{-1}|$. Since $|G_2^{-1}|\neq 0$, $|\Theta(p)|\neq 0$.

Q.E.D.

4-4 STATE EQUATION

In this section, a method of formulating state equations of RCG networks for maximum proper trees is studied.

The characteristic part of the fundamental loop matrix for a maximum proper tree T_{pmax} of an RCG network is assumed to be given as

$$\begin{vmatrix} F_{CC}, F_{C\Sigma} \\ F_{\Sigma C}, F_{\Sigma\Sigma} \end{vmatrix}, \tag{4-4.1}$$

where the subscript Σ represents resistor- and gyrator-branches, that is,

$$F_{C\Sigma}=[F_{CR}, F_{CG}], \quad F_{\Sigma C}=\begin{vmatrix} F_{RC} \\ F_{GC} \end{vmatrix}, \quad F_{\Sigma\Sigma}=\begin{vmatrix} F_{RR}, F_{RG} \\ F_{GR}, F_{GG} \end{vmatrix}. \tag{4-4.2}$$

From the matrix (4-4.1), Kirchhoff's voltage and current laws state,

$$\begin{aligned} -v_{C1}+e_{C1}&=F_{CC}V_{C2}+F_{C\Sigma}v_{\Sigma 2} \\ -V_{\Sigma 1}+e_{\Sigma 1}&=F_{\Sigma C}v_{C2}+F_{\Sigma\Sigma}v_{\Sigma 2} \end{aligned} \tag{4-4.3}$$

and

$$i_{C2}-j_{C2}=F'_{CC}i_{C1}+F'_{\Sigma C}i_{\Sigma 1} \qquad (4\text{-}4.4)$$

$$i_{\Sigma 2}-j_{\Sigma 2}=F'_{C\Sigma}i_{C1}+F'_{\Sigma\Sigma}i_{\ 1},$$

where the subscripts 1 and 2 represent links and twigs, repectively. The vector e. is a loop source voltage vector associated with the subscripts, and j. is a cut-set source current vector associated with the subscripts.

The voltage and current relations of network-elements are given as

$$\begin{vmatrix} i_{C1} \\ i_{C2} \end{vmatrix} = \frac{d}{dt}\begin{vmatrix} C_1, & 0 \\ 0 & ,C_2 \end{vmatrix}\begin{vmatrix} v_{C1} \\ v_{C2} \end{vmatrix}, \qquad (4\text{-}4.5)$$

$$\begin{vmatrix} v_{\Sigma 1} \\ i_{\Sigma 2} \end{vmatrix} = \begin{vmatrix} R_{\Sigma 1}, & 0 \\ 0 & ,G_{\Sigma 2} \end{vmatrix}\begin{vmatrix} i_{\Sigma 1} \\ v_{\Sigma 2} \end{vmatrix}, \qquad (4\text{-}4.6)$$

where

$$R_{\Sigma 1}=\begin{vmatrix} R_1, & 0 \\ 0 & ,G_1 \end{vmatrix}, \qquad G_{\Sigma 2}=\begin{vmatrix} R_2^{-1}, & 0 \\ 0 & ,G_2^{-1} \end{vmatrix}.$$

Eliminating $v_{\Sigma 1}$, $v_{\Sigma 2}$, $i_{\Sigma 1}$, $i_{\Sigma 2}$, i_{C1} and i_{C2} from Eq.(4-4.3)-(4-4.6), we obtain a matric network first-order differential equation as

$$M\frac{d}{dt}\begin{vmatrix} v_{C2} \\ v_{C1} \end{vmatrix} = -N\begin{vmatrix} v_{C2} \\ v_{C1} \end{vmatrix} + L\begin{vmatrix} e_{C1} \\ j_{C2} \\ e_{\Sigma 1} \\ j_{\Sigma 2} \end{vmatrix}, \qquad (4\text{-}4.7)$$

where

$$M=\begin{vmatrix} C_2, & (-F'_{CC}+F'_{\Sigma C}R_0^{-1}F_{\Sigma\Sigma}G_{\Sigma 2}^{-1}F'_{C\Sigma})C_1 \\ 0\ , & F_{C\Sigma}G_0^{-1}F'_{C\Sigma}C_1 \end{vmatrix}$$

$$N=\begin{vmatrix} F'_{\Sigma C}R_0^{-1}F_{\Sigma C} & ,0 \\ F_{CC}-F_{C\Sigma}G_0^{-1}F'_{\Sigma\Sigma}R_{\Sigma 1}^{-1}F_{\Sigma C}, & 1 \end{vmatrix}$$

$$L=\begin{vmatrix} 0,1, & F'_{\Sigma C}R_0^{-1} & ,-F'_{\Sigma C}R_0^{-1}F_{\Sigma\Sigma}G_{\Sigma 2}^{-1} \\ 1,0,-F_{C\Sigma}G_0^{-1}F'_{\Sigma\Sigma}R_{\Sigma 1}^{-1}, & & -F_{C\Sigma}G_0^{-1} \end{vmatrix}$$

$$R_0=R_{\Sigma 1}+F_{\Sigma\Sigma}G_{\Sigma 2}^{-1}F'_{\Sigma\Sigma}, \quad G_0=G_{\Sigma 2}+F'_{\Sigma\Sigma}R_{\Sigma 1}^{-1}F_{\Sigma\Sigma} \tag{4-4.8}$$

If the submatrix of M, $F_{C\Sigma}G_0^{-1}F'_{C\Sigma}C_1$, is a zero matrix, the vector v_{C1} may be eliminated from Eq.(4-4.7), and a state equation may be obtained whose order is equal to the order of complexity given by Theorem 4-3.2. Here let us prove that the matrix $F_{C\Sigma}G_0^{-1}F'_{C\Sigma}C_1$ is a zero matrix.

Lemma 4-4.1

For a maximum proper tree T_{pmax}, the matrix $F_{C\Sigma}G_0^{-1}F'_{C\Sigma}$ is a zero matrix.

Proof: Assume that the matrix $F_{C\Sigma}G_0^{-1}F'_{C\Sigma}$ is a nonzero matrix. There exist i, j, k and l such that

$$f_{C\Sigma}(i,j)g_0^{-1}(j,k)f_{C\Sigma}(l,k)\neq 0 \tag{4-4.9}$$

If $f_{C\Sigma}(i,j)\neq 0$, there exists a link-capacitor-branch (denoted by S_i) corresponding to the row, $f_{C\Sigma}(i)$, and there exists a twig-Σ-branch (denoted by g_j) corresponding to the column, $f_{C\Sigma}\{j\}$. The branch g_j is in the $loop(S_i)$. If $f_{C\Sigma}(l,k)\neq 0$, there exists a link-capacitor-branch (denoted by S_l) corresponding to $f_{C\Sigma}(l)$, and there exists a twig-Σ-branch (denoted by g_k) corresponding to $f_{C\Sigma}\{k\}$. The branch g_k is in the $loop(S_l)$. Then g_j and g_k are gyrator-branch, otherwise, there exists a proper tree which contains more capacitor-branches than T_{pmax}.

Examine the relation between $g_0^{-1}(j,k)\neq 0$ and the network-topology. From the matries (4-4.2) and (4-4.8), the matrix G_0 is written as

$$G_0=\begin{vmatrix}1,0,-F'_{RR},-F'_{GR}\\0,1,-F'_{RG},-F'_{GG}\end{vmatrix}\begin{vmatrix}R_2^{-1}, & 0, & 0, & 0\\0 & ,G_{D2}^{-1}, & 0, & 0\\0 & ,0 & ,R_1^{-1}, & 0\\0 & ,0, & 0 & ,G_{D1}^{-1}\end{vmatrix}\begin{vmatrix}1 & , & 0\\0 & , & E_{D2}\\-F_{RR} & , & -F_{RG}\\-E_{D1}F_{GR} & , & -E_{D1}F_{GG}\end{vmatrix}$$

$$\triangleq Q_{\Sigma}D_{\Sigma}^{-1}Q'_{+\Sigma}, \qquad (4\text{-}4.10)$$

where $E_{D1}=E_1\dotplus\cdots\dotplus E_1$, $E_{D2}=E_1\dotplus\cdots\dotplus E_1$.

The matrices G_{D1} and E_{D1}"G_{D2} and E_{D2}" are of the same dimension. The matrix Q_{Σ} is a fundamental cut-set matrix of $G[T_{pmax}(C);\bar{T}_{pmax}(C)\}$ (denoted by G_{Σ}) for $T_{pmax}(\Sigma)$. The matrix D_{Σ} is diagonal. The matrix $Q_{+\Sigma}$ is a fundamental cut-set matrix of $G^+[T_{pmax}(C);\bar{T}_{pmax}(C)\}$ (denoted by G^+_{Σ}) for $T_{pmax}(\Sigma)$.

Since $g_0^{-1}(j,k)$ is written as

$$g_0^{-1}(j,k)=Cof.G_0(k,j)/|G_0|, \qquad (4\text{-}4.11)$$

if $g_0^{-1}(j,k)\neq 0$, $Cof.G_0(k,j)\neq 0$. The $Cof.G_0(k,j)$ is the determinant of the matrix obtained by omitting the k-th row from Q_{Σ} and the j-th colmn from $Q'_{+\Sigma}$ in (4-4.10), which corresponds to contracting g_k in G_{Σ} and g_j^+ in G^+_{Σ}. Therefore the terms in the expansion of $Cof.G_0(k,j)$ correspond to common trees of $G_{\Sigma}[g_k]$ and $G^+[g_j^+]$. Moreover, there exists at least one independent common tree in $G_{\Sigma}[g_k]$ and $G^+_{\Sigma}[g_j^+]$, which is denoted by T_c.

Since g_k is in the $loop(S_l)$, a set $\{S_l,T_c\}$ is a tree in G_1, where $G_1\triangleq G[T_{pmax}(C);\bar{T}_{pmax}(C)-S_i-S_l\}$. Since g_j is in the $loop(S_i)$, (that is, g_j^+ in the $loop(S_i)$ of G^+), a set $\{S_i,T_c\}$ is a tree in G_1^+.

There are two cases, one of which is that $|T_c-T_c\cap T_c^+|$ is odd, and the other of which is that $|T_c-T_c\cap T_c^+|$ is even.

i) The case where $|T_c-T_c\cap T_c^+|$ is odd.

Suppose that the capacitor-branches S_i and S_l are a pair of

	G_1	G_1^+
	$S_i \leftarrow\rightarrow g_m$	$S_i \leftarrow\rightarrow g_m^+$
	$S_l \leftarrow\rightarrow g_m^+$	$S_l \leftarrow\rightarrow g_m$
Tree	$\{S_l, T_c\}$ $\updownarrow$ $\{g_m^+, T_c\}$	$\{S_i, T_c\}$ $\updownarrow$ $\{g_m^+, T_c\}$

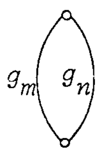
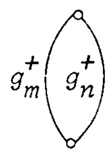
$\leftarrow\rightarrow$ correspondence

Table 4-4.1
Correspondence of capacitors with a pair of gyrator-branches

provisional gyrator-branches g_m and g_m^+ in G_1 and G_1^+, respectively. (See Table 4-4.1.) Then a set $\{g_m^+, T_c\}$ is an independent common tree in G_1 and G_1^+. From Corollary of Lemma 4-3.1, in G_1 there exists a proper tree (denoted by T_{p1}) which contains g_m and g_m^+, that is, S_i and S_l. Consequently a set $\{T_{pmax}(C), T_{p1}\}$ is a proper tree of G, which contains more capacitor-branches than T_{pmax}. This contradicts that T_{pmax} is a maximum proper tree.

ii) The case where $|T_c - T_c \cap T_c^+|$ is even.

Consider the graph which is union of G_1 and the graph which consists of parallel provisional gyrator-branches g_m and g_n.

	G_2		G_2^+	
G_1	$S_i \leftarrow\rightarrow g_m^+$ $S_l \leftarrow\rightarrow g_n^+$	$\cup$ 	G_1^+: $S_i \leftarrow\rightarrow g_m$ $S_l \leftarrow\rightarrow g_n$	$\cup$ 
Tree	$\{S_l, T_c\} \cup g_m$ or g_n $\updownarrow$ $\{g_n^+, T_c\} \cup g_m$		$\{S_i, T_c\} \cup g_m^+$ or g_n^+ $\updownarrow$ $\{g_m, T_c\} \cup g_n^+$	

$\leftarrow\rightarrow$ correspondence

Table 4-4.2 Graph G_2

Suppose that the capacitor-branch S_i is g_m^+ and S_l is g_n^+ in the graph, which is denoted by G_2. Then G_2^+ is union of G_1^+, where S_i is supposed to be g_m and S_l is supposed to be g_n, and the graph which consists of parallel provisional gyrator-branches g_m^+ and g_n^+. A set $\{g_m, g_n^+, T_c\}$ is an independent common tree of G_2 and G_2^+. (See Table 4-4.2.) From Corollary of Lemma 4-3.1, in G_2 there exists a proper tree which contains g_m^+. Consequently there exists a proper tree (denoted by T_{p2}) including S_i in G_1. Then a set $\{T_{pmax}(C), T_{p2}\}$ is a proper tree in G, which contains more capacitor-branches than T_{pmax}. This contradicts that T_{pmax} is a maximum proper tree.

From the above discussion, we obtain

$$F_{C\Sigma}G_0^{-1}F'_{C\Sigma}\equiv 0. \qquad (4\text{-}4.12)$$

Q.E.D.

We obtain

Lemma 4-4.2

The following equality holds.

$A'_{SC}=F'_{CC}-F'_{\Sigma C}R_0^{-1}F_{\Sigma\Sigma}G_{\Sigma 2}^{-1}F'_{C\Sigma}$ for a maximum proper tree T_{pmax},

where $A_{SC}\triangleq F_{CC}-F_{C\Sigma}G_0^{-1}F'_{\Sigma\Sigma}R_1^{-1}F_{\Sigma C}$.

Proof: Let $A_{C\Sigma}$ denote $F_{C\Sigma}G_0^{-1}F'_{\Sigma\Sigma}$. If $A_{C\Sigma}$ is a zero matrix, $A_{SC}=F_{CC}$, then $A'_{SC}=F'_{CC}$.

Assume that $A_{C\Sigma}$ is not a zero matrix. Then there exist i and j such that $a_{C\Sigma}(i,j)\neq 0$. If $a_{C\Sigma}(i,j)\neq 0$, there exist k and l such that

$$f_{C\Sigma}(i,k)g_0^{-1}(k,l)f_{\Sigma\Sigma}(j,l)\neq 0. \qquad (4\text{-}4.13)$$

If the branch corresponding to $f_{\Sigma\Sigma}(j)$ is a resistor-branch, it can be proved that T_{pmax} is not a maximum proper tree in the similar manner in the proof of Lemma 4-4.1. Then this branch is a gyrator-branch. The branch corresponding to $f_{C\Sigma}\{k\}$ is evidently a gyrator-branch.

Assume that $A_{C\Sigma}R_{\Sigma 1}^{-1}F_{\Sigma C}$ is not a zero matrix, then there exist i, j, k, l, m and n such that

$$f_{C\Sigma}(i,k)g_0^{-1}f_{\Sigma\Sigma}(j,l)r_{\Sigma 1}^{-1}(j,m)f_{\Sigma C}(n,m)\neq 0. \tag{4-4.14}$$

The element $r_{\Sigma 1}^{-1}\{m\}$ corresponds to a gyrator-branch since $r_{\Sigma 1}^{-1}(j)$ corresponds to a gyrator-branch. Consequently the branches corresponding to nonzero-elements of $F_{C\Sigma}G_0^{-1}F'_{\Sigma\Sigma}R_{\Sigma 1}^{-1}F_{\Sigma\Sigma}$ are a capacitor-branch and gyrator-branches.

Let $\{A\}_g$ denote the partial matrix concering only gyrator-branches of A. Then we obtain

$$\{G_0^{-1}F'_{\Sigma\Sigma}R_{\Sigma 1}^{-1}\}'_g=\{R_{\Sigma 1}^{-1}F_{\Sigma\Sigma}G_0^{-1}\}_g. \tag{4-4.15}$$

Since G_0 is given as

$$G_0=G_{\Sigma 2}+F'_{\Sigma\Sigma}R_{\Sigma 1}^{-1}F_{\Sigma\Sigma}, \tag{4-4.16}$$

by multiplying Eq.(4-4.16) by $F_{\Sigma\Sigma}G_{\Sigma 2}^{-1}$ from the left, we obtain

$$F_{\Sigma\Sigma}G_{\Sigma 2}^{-1}G_0=R_0R_{\Sigma 1}^{-1}F_{\Sigma\Sigma}. \tag{4-4.17}$$

Then we obtain

$$\{G_0^{-1}F_{\Sigma\Sigma}R_{\Sigma 1}^{-1}\}'_g=\{R_0^{-1}F_{\Sigma\Sigma}G_{\Sigma 2}^{-1}\}_g. \tag{4-4.18}$$

Consequently

$$(F_{C\Sigma}G_0^{-1}F'_{\Sigma\Sigma}R_{\Sigma 1}^{-1}F_{\Sigma C})'=F'_{\Sigma C}R_0^{-1}F_{\Sigma\Sigma}G_{\Sigma 2}^{-1}F'_{C\Sigma}. \tag{4-4.19}$$

Then we obtain

$$A'_{SC}=F'_{CC}-F'_{\Sigma C}R_0^{-1}F_{\Sigma\Sigma}G_{\Sigma 2}^{-1}F_{C\Sigma}. \tag{4-4.20}$$

Q.E.D.

From Lemma 4-4.1, v_{1C} can be eliminated from Eq.(4-4.7). From Lemma 4-4.2, we obtain

$$\mathcal{C}\dot{v}_{C2}=-\mathcal{g}v_{C2}+\mathcal{F}u, \qquad (4\text{-}4.21)$$

where

$$\mathcal{C}=C_2+A'_{SC}C_1A_{SC}$$

$$\mathcal{g}=F'_{\Sigma C}R_0^{-1}F_{\Sigma C}$$

$$\mathcal{F}=[A'_{SC}C_1,\ 1,\ F_{\Sigma C}R_0^{-1},\ -A'_{SC}C_1F_{C\Sigma}G_0^{-1}F'_{\Sigma\Sigma}R_{\Sigma 1}^{-1},\ -F'_{\Sigma C}R_0^{-1}F_{\Sigma\Sigma}G_{\Sigma 2}^{-1},$$
$$-A'_{SC}C_1F_{C\Sigma}G_0^{-1}]$$

$$u=\begin{vmatrix}\dot{e}_{C1}\\ j_{C2}\\ e_{\Sigma 1}\\ \dot{e}_{\Sigma 1}\\ j_{\Sigma 2}\\ \dot{j}_{\Sigma 2}\end{vmatrix}$$

If there do not exist special relations among the network-element-values, the matrix $\mathcal{C}$ is nonsingular and a state equation is obtained as

$$\dot{v}_{C2}=Av_{C2}+Bu, \qquad (4\text{-}4.22)$$

where $A=\mathcal{C}^{-1}\mathcal{g}$, $B=\mathcal{C}^{-1}\mathcal{F}$.

The state equation may have no higher than the first order-derivatives of input functions as similarly as that of an RLCT network may have. Therefore it is concluded that in a passive network, its state equation has no higher than the first order-derivatives of input functions unless there exist special relations among the network-element-values.

Property 4-4.1

The matrix $\mathcal{C}$ is positive definite unless there are no special relations among the network-element-values, however, R_0 is not always positive definite.

Proof: The matrix C is given by

$$C=C_2+A_{SC}^tC_1A_{SC}$$

$$=[1,A_{SC}^t]\begin{vmatrix} C_2, & 0 \\ 0 & ,C_1 \end{vmatrix}\begin{vmatrix} 1 \\ A_{SC} \end{vmatrix},$$

then it is positive definite.

The matrix R_0 is given by

$$R_0=R_{\Sigma 1}+F_{\Sigma\Sigma}R_{\Sigma 2}F_{\Sigma\Sigma}^t$$

$$=[1,F_{\Sigma\Sigma}]\begin{vmatrix} R_\Sigma & , 0 \\ 0 & ,R_\Sigma \end{vmatrix}\begin{vmatrix} 1 \\ F_{\Sigma\Sigma} \end{vmatrix},$$

where the second matrix is not always diagonal. It is sum of matrices which are symmetric positive definite and skew-symmetric.

Q.E.D.

Property 4-4.2

If the state equation has derivatives of input functions, there exists at least one set (denoted by S) of branches containing i) capacitor- and voltage source-branches only, or ii) capacitor- and current source-branches only. The set S is

$$S=B_1\oplus Q_2^+\oplus B_3\oplus\cdots\oplus Q_{2n}^+\oplus B_{2n+1} \qquad \text{for i)}$$

$$S=B_1\oplus Q_2^+\oplus\cdots\oplus Q_{2n}^+, \qquad \text{for ii)}$$

where B_{2i-1}"Q_{2i}" is a loop"cut-set" in G^*.

Proof: See Appendix V.

Q.E.D.

Property 4-4.3

If the matrix g in Eq. (4-4.21) is singular without special relations among the network-element-values, there does not exist a tree in G\{all capacitor-branches\} which is a proper tree of G.

Proof: See Appendix V.

Q.E.D.

4-5 EXAMPLE

Fig. 4-5.1 shows an example for the results in this chapter. Choose a maximum proper tree T_{pmax} in the network as

$$T_{pmax}=\{C_3,C_4,C_5,R_2,R_3,g_2,g_2^+,g_3,g_3^+\}.$$

The matrices G_0^{-1} and R_0^{-1} for T_{pmax} are obtained as

$$G_0^{-1}=\begin{vmatrix} R_2, & 0, & 0, & 0, & 0, & 0 \\ 0, & R_3, & 0, & a_1, & 0 & -a_2 \\ 0, & 0, & 0, & r_2, & 0, & 0 \\ 0, & a_1, & -r_2, & a_3, & -a_4, & -a_5 \\ 0, & 0, & 0, & a_4, & 0, & a_6 \\ 0, & -a_2, & 0, & -a_5, & -a_6, & a_7 \end{vmatrix}$$

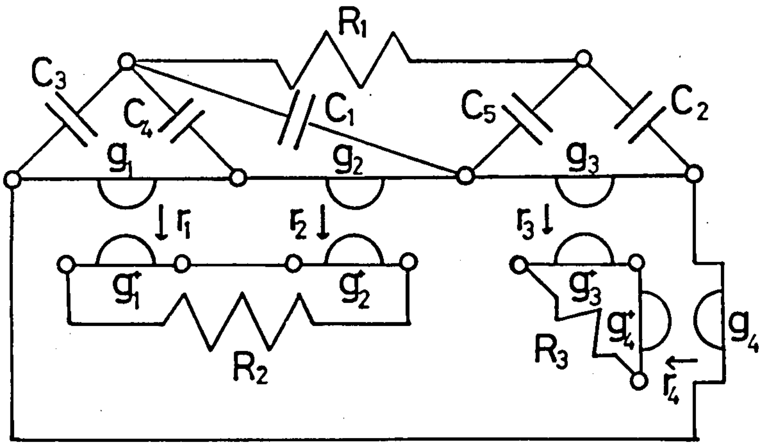


Fig. 4-5.1 Example

$$R_0^{-1}=\frac{1}{R_1 r_2^2 (r_3+r_4)^2}\begin{vmatrix} b_1, & -b_2, & 0, & 0, & 0 \\ -b_2, & b_3, & -b_4, & b_5, & -b_6 \\ 0, & b_4, & 0, & 0, & 0 \\ 0, & b_5, & 0, & b_7, & -b_8 \\ 0, & b_6, & 0, & -b_8, & 0 \end{vmatrix},$$

where

$a_1=R_3 r_2/(r_3+r_4)$ $\qquad a_2=R_3/(r_3^{-1}r_4+1)$

$a_3=\{R_1^{-1}R_3^{-1}(r_3^{-1}+r_4^{-1})^2+r_3^{-2}r_4^{-2}\}/\{R_3^{-1}r_2^{-2}(r_3^{-1}+r_4^{-1})^2\}$

$a_4=r_2/(r_3^{-1}r_4+1)$ $\qquad a_5=R_3 r_2 r_3^{-1} r_4^{-2}/(r_3^{-1}+r_4^{-1})^2$

$a_6=1/(r_3^{-1}+r_4^{-1})$ $\qquad a_7=R_3/(r_3^{-1}r_4+1)$

$b_1=r_1^2(r_3+r_4)^2$ $\qquad b_2=r_1 r_2(r_3+r_4)^2$

$b_3=R_1R_2(r_3+r_4)^2+R_1R_3r_2^2+r_2^2(r_3+r_4)^2$

$b_4=R_1 r_1(r_3+r_4)^2$ $\qquad b_5=R_1R_3r_1r_2$

$b_6=R_1r_1r_2(r_3+r_4)$ $\qquad b_7=R_1R_3r_1^2$

$b_8=R_1r_1^2(r_3+r_4)$.

Since $F_{C\Sigma}$ is given as

$$F_{C\Sigma}=\begin{vmatrix} 0,0,1,0,0,0 \\ 0,0,0,0,1,0 \end{vmatrix},$$

it is ascertained that Lemma 4-4.1, that is, $F_{C\Sigma}G_0^{-1}F'_{C\Sigma}\equiv 0$ is true.

Since $F_{\Sigma C}$ is given as

$$F_{\Sigma C}=\begin{vmatrix} 0, & 1, & 1 \\ 1, & 1, & 0 \\ 0, & 0, & 0 \\ -1, & -1, & 0 \\ 0, & 0, & 0 \end{vmatrix},$$

the matrix g is given as

$$g=\begin{vmatrix} b_3-2b_5+b_7 & , -b_2+b_3-2b_5+b_7 & , -b_2 \\ -b_2+b_3-2b_5+b_7 & , b_1-2b_2+b_3-2b_5+b_7 & , b_1-b_2 \\ -b_2 & , b_1-b_2 & , b_1 \end{vmatrix},$$

it is singular. This ascertains that Property 4-4.3 holds.

The network has a unique solution if and only if $r_3+r_4 \neq 0$.

The matrix $\boldsymbol{C}$ is given as

$$\boldsymbol{C}=\begin{vmatrix} C_3+d_1^2C_1+d_3^2C_2, & d_1d_2C_1+d_3^2C_2, & d_3C_2 \\ d_1d_2C_1+d_3^2C_2 & , C_4+d_2^2C_1+d_3^2C_2, & d_3C_2 \\ d_3C_2 & , d_3C_2 & , C_5+C_2 \end{vmatrix},$$

where

$$d_1=r_1^{-1}r_2, \quad d_2=d_1-1, \quad d_3=d_2/(r_3^{-1}r_4+1).$$

The matrices $\boldsymbol{C}$ and $\boldsymbol{g}$ lead to the matrix A of the state equation.

Fig.4-5.2 shows an example of networks which have no proper tree. The voltage across"the current through" j-gyrator-port is denoted by v_j"i_j". The voltage- and current-relations of the gyrators are

$$\begin{vmatrix} v_1 \\ v_1' \end{vmatrix}=\begin{vmatrix} 0 & , r_1 \\ -r_1, & 0 \end{vmatrix}\begin{vmatrix} i_1 \\ i_1' \end{vmatrix}, \quad \begin{vmatrix} v_2 \\ v_2' \end{vmatrix}=\begin{vmatrix} 0 & , r_2 \\ -r_2, & 0 \end{vmatrix}\begin{vmatrix} i_2 \\ i_2' \end{vmatrix}, \quad \begin{vmatrix} v_3 \\ v_3' \end{vmatrix}=\begin{vmatrix} 0 & , r_3 \\ -r_3, & 0 \end{vmatrix}\begin{vmatrix} i_3 \\ i_3' \end{vmatrix}.$$

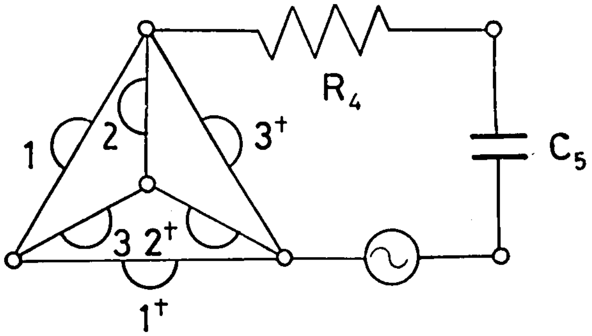


Fig.4-5.2 Example of network without proper tree.

Then the following equalities hold.

$v_3=R_3 i_3'$, $i_3'=i_1+i_1'$, $v_3=v_1-v_1'=R_1(i_1+i_1')$.

Then $R_3=R_1$. Similarly $R_3=-R_2$ is obtained.

Consequently even if $R_3=R_1=-R_2$, or the gyrators 2 and 3 may be removed, the current are indeterminate.

It can be said that the network 1) has no physical meaning or 2) does not satisfy the Kirchhoff's laws.

4-6 CONCLUDING REMARKS

The conjectures[6] given by E.S.Kuh and *et.al.* for the solvability and the order of complexity of RCG networks are proved in Section 4-3.1. A necessary and sufficient condition for the solvability is obtained in Section 4-3.2. It depend on the network-topology and resistance values of the gyrators. Moreover a sufficient condition for the solvability depending on the network-topology is obtained in Section 4-3.2. An explicit form of state equations, whose state variables are voltages across the twig-capacitors of a maximum proper tree, is obtained in Section 4-4. It is of the same form as state equations of networks satisfying special assumption given by J.Tow[15].

The concept '*independent common tree*' is given in order to obtain the above results. It is obtained from an algebraic point of view. Let us show the concept from a graphical point of view. Consider a common tree T_c of G and G^+. Let K_2 be $T_c-T_c\cap T_c^+$. From Property 4-3.2, if $|K_2|$ is odd, T_c is not an independent common tree. Assume that $|K_2|$ is even and that $K_2 \triangleq \{a_1,\cdots,a_{2n}\}$, where a_i is a gyrator-branch. Let $b_i^{(k)}$ denote a_i or a_i^+. Then consider 2^{2n} sets as $S_k=\{b_1^{(k)},\cdots,b_{2n}^{(k)}\}$, where $k=1,\cdots,2^{2n}$. Let $\mathcal{S}$ be $\{S_1, \quad ,S_{2^{2n}}\}$. Let $\mathcal{S}_1$ denote a subset of $\mathcal{S}$ whose elements are common trees of G_s and G_s^+, where $G_s=G[T_c-K_2;\bar{T}_c-K_2^+\}$. A set {one element of $\mathcal{S}_1$} $\{T_c \ T_c^+\}$ is a common tree of the same class of T_c. The number of elements in $\mathcal{S}_1$ which contain even"odd" number of mark $^+$ is denoted by c"d". If $c\neq d$, the common tree T_c is an independent common tree.

The above results are based on the concept '*proper tree*'. A physical meaning of the concept is as follows. Consider a network equation one of whose unknown variables is voltage across"current thruogh" one port of a gyrator. It is necessary that current thruogh"voltage across" the other port of the gyrator be determined from the network-topology. Considering this from a graphical point of view, it is necessary that the branch corresponding to one port of the gyrator is a link"twig" and that to the other branch is also a link"twig".

CHAPTER 5 ACTIVE NETWORK(4)-(8)

5-1 INTRODUCTION

In the preceding chapters, the solvability, the order of complexity and explicit forms of state equations in passive networks containing passive elements, that is, resistors, inductors, capacitors, ideal-transformers and gyrators, are studied. The networks considered in this chapter contain both passive elements and active elements such as transistors and electric tubes. It is well-known that in the linear operations, a network containing active elements is equivalent to a network containing dependent voltage-sources and dependent current-sources, the voltages and the currents of which depend on one or more of the voltages across or currents through the network-elements.

Several papers[18]-[21] discussing on the problems of the solvability and the order of complexity of active networks has been published. J.Tow[18] has introduced two graphs G_V and G_I obtained from the original network; the graph G_V specifies the voltage relations of the network, and the graph G_I specifies the current relations of the network. He has derived a condition for the solvability and the order of complexity by means of the above two graphs G_V and G_I. E.J.Purslow[19][20] has approached the problems from an algebraic point of view. T.Ozawa[21] has considered, by means of the two graph-method, the problems for networks whose topology is of wider varieties than that considered by J.Tow. From a topological point of view, it can be said that the two graph-method has given the solutions of the problems of solvability and the order of complexity.

Several papers[22]-[26] have discussed on formulating state equations for active networks. In the network considered in the previous chapters, the network-elements corresponding to state variables can be determined from their network-topology only. Since the state equations are obtained by an algebraic method, in the above formulations of state equations of active networks, the network-elements corresponding to state variables and the degree of the state equations obtained are not determined until the state equations are obtained.

There are few studies[20] on network-equations without unique solutions.

Here a formulation of state equations for networks with unique solutions and canonical forms of network-equations without unique solutions are considered from the network-topological point of view.

In Section 5-2, the dependent sources and network toplogy in active networks to be considered are described. In Section 5-3, the results with respect to the problems of the solvability and the order of complexity are described. In Section 5-4, the state equations, the formulation of which depends strongly on the topological condition for the network-solvability and on the order of complexity are studied. In Section 5-5, a canonical form of network-equations without unique solutions are considered by means of matrix pencil.

5-2 ACTIVE NETWORK ELEMENTS AND GRAPHS

5-2.1 ACTIVE NETWORK ELEMENT

There are four types of dependent sources, the voltage- and current-relations of which are given, respectively, by

$$\begin{vmatrix} v_o \\ i_i \end{vmatrix} = \begin{vmatrix} \mu, 0 \\ 0, 0 \end{vmatrix} \begin{vmatrix} v_i \\ i_o \end{vmatrix} \tag{5-2.1}$$

$$\begin{vmatrix} v_o \\ v_i \end{vmatrix} = \begin{vmatrix} 0, r \\ 0, 0 \end{vmatrix} \begin{vmatrix} i_o \\ i_i \end{vmatrix} \tag{5-2.2}$$

$$\begin{vmatrix} i_o \\ v_i \end{vmatrix} = \begin{vmatrix} 0, h \\ 0, 0 \end{vmatrix} \begin{vmatrix} v_o \\ i_i \end{vmatrix} \tag{5-2.3}$$

$$\begin{vmatrix} i_o \\ i_i \end{vmatrix} = \begin{vmatrix} 0, g \\ 0, 0 \end{vmatrix} \begin{vmatrix} v_o \\ v_i \end{vmatrix}, \tag{5-2.4}$$

where the subscripts o and i signify out-put and in-put, respectively.

The network-elements specified by Eq,(5-2.1)-(5-2.4) are called voltage-controlled voltage source, current-controlled voltage source, current-controlled current source and voltage-controlled current source, respectively.

A current-"voltage-"controlled current"voltage" source can be replaced by the cascade connection of a current-"voltage-"controlled voltage"current" source and a voltage-"current-"controlled current"voltage" source. Active networks considered here may be assumed to contain only current-controlled voltage sources and voltage-controlled current source but no other dependent sources. Furthermore, network-elements such as ideal transformers, gyrators and negative-impedance convertors which appear in general active networks can be replaced by combinations of the dependent sources. Inductors can be replaced by capacitors and dependent sources.

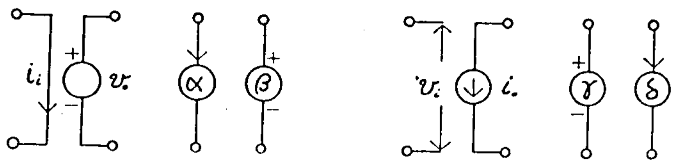
Resistors can be replaced by the dependent sources. Therefore,

Remark

The network considered in this chapter contains only capacitors, independent sources, current-controlled voltage sources and voltage-controlled current sources. The number of these elements are n_C, n_β and n_δ, respectively, and let n_D and n_e be $n_\beta+n_\delta$ and n_C+n_D, respectively.

A current-controlled voltage source and a voltage-controlled current source are symbolically shown in Fig.5-2.1(a) and (c). Their expressions used here are shown in Fig.5-2.1(b) and (d), where α, β, γ and δ express a current sensor, a controlled voltage sources, a voltage sensor and a controlled current source, respectively. From this, a current-controlled voltage source"a voltage-controlled current source" may be considered as a voltage source controlled by current through a current sensor"a current source controlled by voltage across a voltage sensor".

The voltage and current relation of controlled sources and sensors is given by



(a) (b) (c) (d)

Fig.5-2.1

(a) a current-controlled voltage source and (b) its expression

(c) a voltage-controlled current source and (d) its expression

$$H_{VD}v_D = H_{ID}i_D, \tag{5-2.5}$$

where

$$v_D = \begin{vmatrix} v_\beta \\ v_\gamma \end{vmatrix}, \qquad i_D = \begin{vmatrix} i_\alpha \\ i_\delta \end{vmatrix},$$

$$H_{VD} = \begin{vmatrix} 1, & 0 \\ 0, & G_\gamma \end{vmatrix}, \qquad H_{ID} = \begin{vmatrix} R_\alpha, & 0 \\ 0 & ,1 \end{vmatrix}.$$

The vectors v_β and v_γ consist of voltages across controlled voltage sources and voltage sensors, respectively. The vectors i_α and i_δ consist of currents through current sensors and controlled current sources, respectively. The matrices R_α and G_γ are diagonal ones of the dimension n_β and n_δ, respectively.

5-2.2 ACTIVE NETWORKS AND GRAPHS

5-2.2.1 VOLTAGE AND CURRENT GRAPHS

Let graph G^* correspond to an active network N^* which contains capacitors, current sensors, controlled voltage sources, voltage sensors, controlled current sources and independent sources. Graph G is obtained from G^* by contracting independent voltage sources and deleting independent current sources.

Now, two graphs are introduced, which are the voltage and the current graphs as denoted by G_V and G_I, respectively. The voltage graph G_V is obtained from G by contracting all the current sensors α and deleting all the current sources δ. The current graph G_I is obtained from G by deleting all the voltage sensors γ and contracting all the voltage sources β. The graphs G_V and G_I

represent the voltage and current restrictions of the network.

Every branch of G_V and G_I has one to one correspondence to a branch of each other, that is, a capacitor-branch C in G_V to the capacitor-branch C in G_I, a voltage source-branch β in G_V to its current sensor-branch α in G_I, and a voltage sensor-branch γ in G_V to its current source-branch δ in G_I. Every branch in G_V and its corresponding branch in G_I are assumed to have the same label.

Let B_V and Q_I denote a basic loop matrix of G_V and a basic cut-set matrix of G_I, respectively. Then the Kirchhoff's voltage and current laws state

$$B_V v=e \qquad (5\text{-}2.6)$$

$$Q_I i=j, \qquad (5\text{-}2.7)$$

where

$$v=\begin{vmatrix} v_C \\ v_D \end{vmatrix}, \qquad i=\begin{vmatrix} i_C \\ i_D \end{vmatrix}.$$

Partition B_V and Q_I as

$$B_V=[B_{VC},B_{VD}], \qquad (5\text{-}2.8)$$

$$Q_I=[Q_{IC},Q_{VD}]. \qquad (5\text{-}2.9)$$

Then Eq.(5-2.6) and (5-2.7) are rewritten, respectively, as

$$[B_{VC},B_{VD}]\begin{vmatrix} v_C \\ v_D \end{vmatrix}=e, \qquad [Q_{IC},Q_{ID}]\begin{vmatrix} i_C \\ i_D \end{vmatrix}=j. \qquad (5\text{-}2.10)$$

Definition 5-2.1

Let T_V and T_I denote trees of G_V and G_I, respectively. A tree-pair of T_V and T_I such that $|T_V \cap T_I|$ is maximum is called a maximum tree-pair of G_V and G_I @, and is denoted by $\{T_V:T_I\}$.

@It is called MCT in [10].

Definition 5-2.2

If $T_V = T_I$, T_c is called a common tree of G_V and G_I, where $T_V = T_I \triangleq T_c$.

Definition 5-2.3

A maximum tree-pair $\{T_V : T_I\}$ such that $|T_V(C) \cap T_I(C)|$ is maximum "minimum" is called a normal"primitive" maximum tree-pair, which is denoted by $\{T_V^N : T_I^N\}$"$\{T_V^P : T_I^P\}$".

Definition 5-2.4

A common tree T_c such that $|T_c(C)|$ is maximum is called a normal common tree, which is denoted by T_c^N.

5-2.2.2 INDUCED DIGRAPH

In Section 5-2.2.1, two graph G_V and G_I are introduced. Here let us introduce a directed graph or digraph induced from G_V and G_I, which specifies voltage and current relation.

Let $t_i\{i=1,2,\cdots\}$ and $l_j\{j=1,2,\cdots\}$ denote twigs and links, respectively, for a common tree T_c.

Definition 5-2.5

A digraph obtained by the following procedures is called an induced digraph of G_V and G_I for a common tree T_c, which is denoted by $G^{\#}$.

1) Every branch of G_V and G_I corresponds to a node of $G^{\#}$.

2) There exists a directed branch in $G^{\#}$ from a node corresponding to a link l_i (denoted by $V(l_i)$) to a node corresponding to a twig t_j (denoted by $V(t_j)$), where t_j is in the $loop(l_i)$ in G_V.

3) There exists a directed branch in $G^{\#}$ from a node $V(t_i)$ to a node $V(l_j)$, where l_j is in the $cut\text{-}set(t_i)$ in G_I.

An example for $G^{\#}$ is shown in Fig.5-2.2. For the network shown in Fig.5-2.2(a), its voltage and its current graphs are shown in Fig.5-2.2(b) and (c), respectively. Its induced graph for a common tree $T_c=\{1,C_2,C_4\}$ is shown in Fig.5-2.2(d).

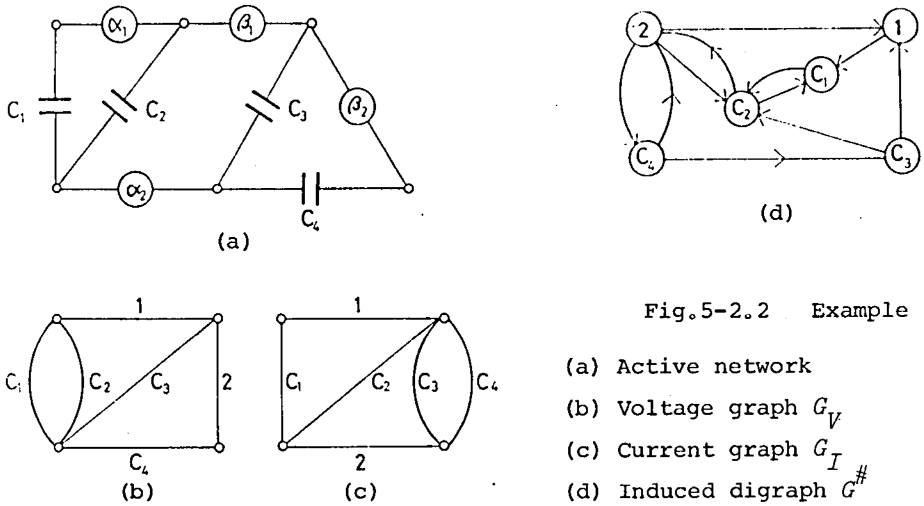


Fig.5-2.2 Example

(a) Active network

(b) Voltage graph G_V

(c) Current graph G_I

(d) Induced digraph $G^{\#}$

From Definition 5-2.5, some properties for $G^{\#}$ are obtained.

Property 5-2.1

The induced digraph of G_V[*some twigs* $t_i\{i=1,\cdots\}$*;some links* l_i $\{j=1,\cdots\}$} and G_I[*the twigs;the links*} is obtained by deleting nodes[@] $V(t_i)$ and $V(l_j)$ from $G^{\#}$.

@ 'Deleting nodes' means that the nodes and the directed branch touching the nodes are removed.

The directed branch from V_i to V_j in $G^{\#}$ is denoted by $d(V_i,V_j)$, where if V_i corresponds to a twig"link", V_j corresponds to a link "twig".

The directed loop in $G^{\#}$ containing $d(V_1,V_2)$, $d(V_2,V_3)$, $\cdots\cdots$, $d(V_{n-1},V_n)$ and $d(V_n,V_1)$ is denoted by $diloop\{V_1,V_2, \quad ,V_n,V_1\}$. There are always diloops if there are neither self loop nor bridge in G_V and G_I. If the node V_i corresponds to a twig"link", the node V_{i+1} corresponds to a link"twig" in the $diloop\{V_1,V_2,\cdots,V_n,V_1\}$. Therefore the number of the branches"nodes" in the $diloop$ is even.

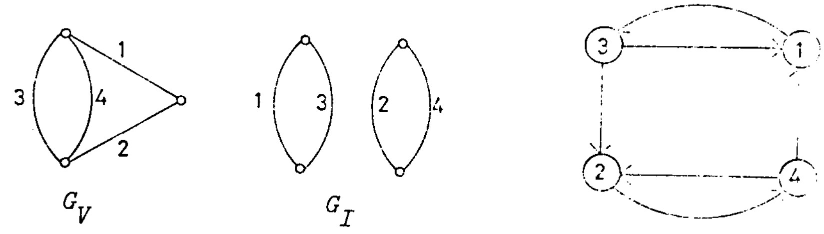
The tree-transformation in G_V and G_I may be considered, such as $T_c-t_1+l_1$ is a tree in G_V, $T_c-t_1+l_2$ is a tree in G_I, $T_c-(t_1+t_2)+(l_1+l_2)$ is a tree in G_V, $T_c-(t_1+t_2)+(l_2+l_3)$ is a tree in G_I, and so on.

If another common tree $T_{c1}=T_c-(t_1+\cdots+t_n)+(l_1+\cdots+l_n)$ can be obtained by transforming links l_i"twigs t_i" $\{i=1,\cdots,n\}$ to twigs "links" in G_V and l_{i+1}"t_i" {where $n+1\underline{\Delta}1$} to twigs"links" in G_I, there exists a directed loop containing nodes $V(l_i)$ and $V(t_i)$ in $G^{\#}$, because t_i is in the $loop(l_i)$ in G_V and l_{i+1} is in the $cut\text{-}set(t_i)$ in G_I. This proves

Property 5-2.2

If another common tree T_{c1} represented by $T_{c1}=T_c-(t_1+\cdots t_n)+(l_1+\cdots+l_n)$ can be obtained from T_c, then there exists a directed loop represented by $diloop\{V(l_1),V(t_1),\cdots,V(l_n),V(t_n),V(l_1)\}$ in $G^{\#}$.

The converse of Property 5-2.2 is not always true. Consider



(a) (b) $G^{\#}$

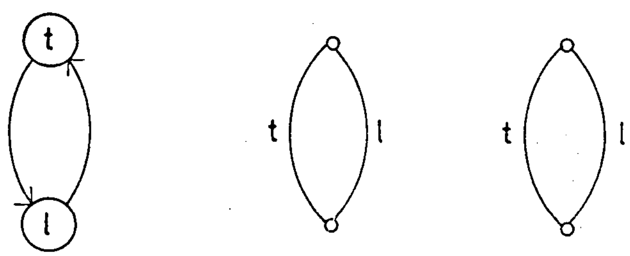
Fig.5-2.3 Example for G_V, G_I and $G^{\#}$.

this in an example shown in Fig.5-2.3. A common tree is assumed to be $T_c=\{1,2\}$ in the graphs G_V and G_I shown in Fig.5-2.3(a). Then the induced digraph $G^{\#}$ for T_c is obtained as shown in Fig.5-2.3(b). There exists a $diloop\{V(3),V(1),V(3)\}$ and set$\{3,2\}$ is a common tree. However, there exists a $diloop\{V(4),V(1),V(3),V(2),V(4)\}$, but set$\{3,4\}$ is not a common tree.

Property 5-2.3

If another common tree represented by $T_{c1}=T_c-t+l$ can be obtained from T_c, there exists a $diloop\{V(t),V(l),V(t)\}$ in $G^{\#}$ for T_c, and the converse is true.

Proof: Consider the subgraphs $G_V[T_c-t;\bar{T}_c-l\}$ and $G_I[T_c-t;\bar{T}_c-l\}$, which are denoted by $\tilde{G}_V$ and $\tilde{G}_I$, respectively. The induced sub-



(a) $G^{\#}$ G_V (b) G_I

Fig.5-2.4 Example for the proof of Property 5-2.3

digraph (denoted by $\widetilde{G}^{\#}$) of G_V and G_I for a common tree $\{t\}$ is shown in Fig.5-2.4(a), then G_V and G_I are the graphes shown in Fig.5-2.4(b). Then the converse is true.

Q.E.D.

Consider the case where there exists a $diloop\{V(l_1),V(t_1),\cdots,V(l_n),V(t_n),V(l_1)\}$ *in* $G^{\#}$ for T_c, but $T_c-(t_1+t_2+\cdots+t_n)+(l_1+l_2+\cdots+l_n)$ is not a common tree. Let two graphs $\widetilde{G}_V$ and $\widetilde{G}_I$ be $G_V[T_c-(t_1+\cdots+t_n);\bar{T}_c-(l_1+\cdots+l_n)\}$ and $G_I[T_c-(t_1+\cdots+t_n);\bar{T}_c-(l_1+\cdots+l_n)\}$, respectively. The induced directed graph of $\widetilde{G}_V$ and $\widetilde{G}_I$ for a common tree $\{t_1,\cdots,t_n\}$ is denoted by $\widetilde{G}^{\#}$. Suppose that $T_c-(t_1+\cdots+t_k)+(l_1+\cdots+l_k)$ is not a tree in $\widetilde{G}_V$ or $T_c-(t_1+\cdots+t_k)+(l_2+\cdots+l_{k+1})$ is not a common tree in $\widetilde{G}_I$ for a certain minimum integer k ($\leq n$). Then there exists at least one loop consisting only of l_k and a subset of $\{l_1,\cdots,l_{k-1}\}$ either in $\widetilde{G}_V$, or of l_{k+1} and a subset of $\{l_2,l_3,\cdots,l_k\}$ in $\widetilde{G}_I$. Assume that there exists such a loop in $\widetilde{G}_V$. Let $\{l_{i_1},\quad,l_{i_m}\}$ be the loop$\{k\geq m\geq 2\}$. Then there exist corresponding twigs t_{i_j} $\{j=1,\cdots,m\}$, each fundamental cut-set for which contains more than one links of $\{l_{i_1},\cdots,l_{i_m}\}$ in $\widetilde{G}_V$. Conversely each link l_{i_j} $\{j=1,\cdots,m\}$ is contained in more than one cut-sets of $cut\text{-}set(t_{i_j})$ in G_V (that is, each $loop(l_{i_j})$ contains more than one twigs t_{i_j}.). Then in $\widetilde{G}^{\#}$, there exist diloops which consist only some nodes of $\{V(l_1),V(t_1),\cdots,V(l_n),V(t_n)\}$. If $m=2$, there may exist $diloop\{V(l_{i_k}),V(t_i),V(l_{i+1}),\cdots,V(t_{i_{k-1}}),V(l_i)\}$ and $diloop\{V(l_i),V(t_{i_k}),V(l_{i_{k+1}}),\cdots,V(t_{i-1}),V(l_i)\}$ in $G^{\#}$. The number of such diloops is more than $m-1$. The nodes in such diloops cover the nodes in the $diloop\{V(l_1),V(t_1),\cdots,V(l_n),V(t_n),V(l_1)\}$. If there exists a loop in $\widetilde{G}_I$, the similar argument holds.

Then we obtain

Property 5-2.4

If there exists a $diloop\{V(l_1),V(t_1),\cdots,V(l_n),V(t_n),V(l_1)\}$ (denoted by D_L) in $G^{\#}$ of G_V and G_I for a common tree T_c, but if the set, $\{T_c-(t_1+\cdots+t_n)+(l_1+\cdots+l_n)\}$, is not a common tree of G_V and G_I, then there exist diloops which consists only of some nodes in D_L, furthermore the nodes in all such diloops cover the nodes in D_L.

From Property 5-2.2-4, the next property holds.

Property 5-2.5

If, in $G^{\#}$ for T_c, there exists a $diloop\{V(l_1),V(t_1),\cdots,$ $V(l_n),V(t_n),V(l_n)\}$ which contains no sub-diloop, then $T_c-(t_1+$ $+t_n)+(l_1+\quad+l_n)$ is a common tree.

5-3 SOLVABILITY AND THE ORDER OF COMPLEXITY

Consider the solvability and the order of complexity for the networks satisfying Assumption A in Chapter 2.
From Eq.(5-2.5) and (5-2.10), a network equation of an active network is given by

$$\theta(p)\begin{vmatrix} v \\ i \end{vmatrix} \triangleq \begin{vmatrix} Cp, & 0, & -1, & 0 \\ 0, & H_{VD}, & 0, & -H_{ID} \\ B_{VC}, & B_{VD}, & 0, & 0 \\ 0, & 0, & Q_{IC}, & Q_{ID} \end{vmatrix} \begin{vmatrix} v_C \\ v_D \\ i_C \\ i_D \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ e \\ j \end{vmatrix}, \qquad (5\text{-}3.1)$$

If and only if the coefficient matrix $\boldsymbol{\theta}(p)$ is nonsigular, the network has a unique solution.

We have

Theorem 5-3.1[18][21]

If a network has a unique solution, there exists a common tree in its voltage and current graphs.

The converse of the theorem is true unless there are no special relations among the network-element-values. It can be said that Theorem 5-3.1 is a necessary and sufficient condition for the topological solvability.

The concept '*common tree*' is given from the relations between network-topology and nonsingular $\theta(p)$. In other words, since the current through a twig is given by the currents through some of the links in G_I, and the voltage across a link is given by the voltages across some of the twigs in G_V, the voltage across" the current through" a network-element" can be given by the current through"the voltage across" the network-element if there exists a common tree in G_V and G_I.

For the order of complexity, we have

Theorem 5-3.2[18][21]

The order of complexity of an active network satisfying the remark in Section 5-3.2 is not greater than $|T_c^N(C)|$.

5-4 STATE EQUATION

Several papers[20][22][24][25] discussing on the formulations of the state equations for linear active networks have been published. The outlines of the formulations are as follows;

Step 1. Obtain a set of network-equations by applying Kirchhoff's voltage and current laws and from the voltage-current relations of the network-element.

Step 2. By eliminating voltage- and current-variables of the non-reactive network-elements, obtain a differential equation in the form of

$$A\dot{x}=Bx+Cu, \tag{5-4.1}$$

where x is a vector whose elements are the voltages across the capacitors and the current through the inductors.

Step 3. Set $i=0$, and $A_0=A$, $B_0=B$, $C_{0,0}=C$, $x_0=x$, $\overset{(0)}{u}=u$, then Eq.(5-4.1) becomes

$$A_i\dot{x}_i=B_ix_i+\sum_{k=0}^{i}C_{i,k}\overset{(k)}{u}. \tag{5-4.2}$$

Step 4. If A_i is nonsingular, go to Step 5. If A_i is singular, transform Eq.(5-4.2) into the following form;

$$\begin{vmatrix} A_{i,1}, A_{i,2} \\ 0 \quad , \quad 0 \end{vmatrix} \begin{vmatrix} \dot{x}_{i,1} \\ \dot{x}_{i,2} \end{vmatrix} = \begin{vmatrix} B_{i,1,1}, B_{i,1,2} \\ B_{i,2,1}, B_{i,2,2} \end{vmatrix} \begin{vmatrix} x_{i,1} \\ x_{i,2} \end{vmatrix} + \sum_{k=0}^{i}C_{i,k}\overset{(k)}{u}. \tag{5-4.3}$$

Then by eliminating $x_{i,2}$ from Eq.(5-4.3), obtain

$$A_{i+1}\dot{x}_{i,1}=B_{i+1}x_{i,1}+\sum_{k=0}^{i+1}C_{i+1,k}\overset{(k)}{u}.$$

Set $i=i+1$, $x_{i+1}=x_{i,1}$, and go to Step 4.

Step 5. Obtain a state equation as

$$\dot{x}_i=A_i^{-1}B_ix_i+A_i^{-1}\sum_{k=0}^{i}C_{i,k}\overset{(k)}{u}. \tag{5-4.4}$$

For the network considered in the preceding chapters, the state equations can be formulated by use of the trees given the order of complexity if the topological solvability conditions are satisfied. The state variables and the order of the formulated state equations are determined from the trees. However, the state variables and the order of the state equations derived by the above formulation for active networks cannot be determined until the state equations are obtained. One reason for this is that the formulation depends on only algebraic operations.

In this section, a formulation of state equations for the networks, which satisfy the following Assumption 5-4.1, is given. This formulation is based on the network-topological condition. The state variables can be determined before the state equations are obtained.

Assumption 5-4.1

The network-solvability, the order of complexity and the rank of matrices which appear in the process of deriving the state equations depend on the network-topology only.

The networks considered here contain only capacitors, current-controlled voltage sources, voltage-controlled current sources and independent sources as mentioned in Section 5-2.

By using the fundamental loop matrix of G_V and the fundamental cut-set matrix of G_I for a normal common tree T_c^N, the Kirchhoff's voltage and current laws lead to

$$\begin{vmatrix} v_S \\ v_{\beta_1} \\ v_{\gamma_1} \end{vmatrix} = \left|\begin{array}{c|cc} B_{SC}, & B_{S\beta_2}, & B_{S\gamma_2} \\ \hline B_{\beta_1 C}, & B_{\beta_1\beta_2}, & B_{\beta_1\gamma_2} \\ B_{\gamma_1 C}, & B_{\gamma_1\beta_2}, & B_{\gamma_1\gamma_2} \end{array}\right| \begin{vmatrix} v_C \\ v_{\beta_2} \\ v_{\gamma_2} \end{vmatrix} + \begin{vmatrix} e_S \\ e_{\beta_1} \\ e_{\gamma_1} \end{vmatrix}, \tag{5-4.5}$$

$$\begin{vmatrix} i_C \\ i_{\alpha_2} \\ i_{\delta_2} \end{vmatrix} = \left|\begin{array}{c|cc} B_{CS}, & B_{C\alpha_1}, & B_{C\delta_1} \\ \hline B_{\alpha_2 S}, & B_{\alpha_2\alpha_1}, & B_{\alpha_2\delta_1} \\ B_{\delta_2}, & B_{\delta_2\alpha_1}, & B_{\delta_2\delta_1} \end{array}\right| \begin{vmatrix} i_S \\ i_{\alpha_1} \\ i_{\delta_1} \end{vmatrix} + \begin{vmatrix} j_C \\ j_{\alpha_2} \\ j_{\delta_2} \end{vmatrix}, \tag{5-4.6}$$

where $B_{\circ\circ}$ and $Q_{\circ\circ}$ are the submatrices of the characteristic parts of the fundamental loop matrix of G_V and the fundamental cut-set matrix of G_I, respectively. The subscript $_1$"$_2$" denotes link-"twig-"dependent sources. The subscript S"C" denotes link-"twig-"capacitors.

The voltage and current relations of dependent sources are given by

$$\begin{vmatrix} v_{\beta_1} \\ v_{\beta_2} \end{vmatrix} = \begin{vmatrix} R_{\alpha_1}, & 0 \\ 0, & R_{\alpha_2} \end{vmatrix} \begin{vmatrix} i_{\alpha_1} \\ i_{\alpha_2} \end{vmatrix}, \tag{5-4.7}$$

$$\begin{vmatrix} i_{\delta_1} \\ i_{\delta_2} \end{vmatrix} = \begin{vmatrix} G_{\gamma_1}, & 0 \\ 0_2, & G_{\gamma_2} \end{vmatrix} \begin{vmatrix} v_{\gamma_1} \\ v_{\gamma_2} \end{vmatrix}, \tag{5-4.8}$$

where the submatrices $R_\circ$ and $G_\circ$ are diagonal.

Eliminating the voltages across and the currents through the dependent sources from Eq.(5-4.5)-(5-4.8), we obtain

$$\begin{vmatrix} 1, & -F \\ 0, & B \end{vmatrix} \begin{vmatrix} i_C \\ i_S \end{vmatrix} = \begin{vmatrix} D, & 0 \\ -A, & 1 \end{vmatrix} \begin{vmatrix} v_C \\ v_S \end{vmatrix} + \begin{vmatrix} \grave{e}_C \\ \tilde{e}_S \end{vmatrix} + \begin{vmatrix} \tilde{j}_C \\ \tilde{j}_S \end{vmatrix}, \tag{5-4.9}$$

where

$$A = B_{SC} + B_{S2}\mathcal{g}^{-1}Q_{21}R_1^{-1}B_{1C}$$

$$B = B_{S2}\mathcal{g}^{-1}Q_{2S}$$

$$D = Q_{C1}\boldsymbol{R}^{-1}B_{1C}$$

$$F = Q_{CS} + Q_{C1}\boldsymbol{R}^{-1}B_{12}G_2^{-1}Q_{2S}$$

$$\tilde{e}_C = Q_{C1}\boldsymbol{R}^{-1}e_1$$
$$\tilde{j}_C = Q_{C1}\boldsymbol{R}^{-1}B_{12}G_2^{-1}j_2 + j_C$$
$$\tilde{e}_S = B_{S2}\boldsymbol{g}^{-1}Q_{21}R_1^{-1}e_1 + e_S$$
$$\tilde{j}_S = B_{S2}\boldsymbol{g}^{-1}j_2$$
$$\boldsymbol{g} = G_2 - Q_{21}R_1^{-1}B_{12}, \qquad \boldsymbol{R} = R_1 - B_{12}G_2^{-1}Q_{12}$$
$$G_2 = \begin{vmatrix} R_{\alpha_2}^{-1}, & 0 \\ 0 & , G_{\gamma_2} \end{vmatrix}, \qquad R_1 = \begin{vmatrix} R_{\alpha_1}, & 0 \\ 0 & , G_{\delta_1}^{-1} \end{vmatrix}$$

$$B_{S2} = \begin{array}{cc} (\beta_2) & (\gamma_2) \\ {[B_{S\beta_2},} & {B_{S\gamma_2}]} \end{array}(S) \qquad B_{1C} = \begin{array}{c} (C) \\ \begin{vmatrix} B_{\beta_1 C} \\ B_{\gamma_1 C} \end{vmatrix} \end{array}\begin{array}{c} \\ (\beta_1) \\ (\gamma_1) \end{array}$$

$$B_{12} = \begin{array}{c} \begin{array}{cc} (\beta_2) & (\gamma_2) \end{array} \\ \begin{vmatrix} B_{\beta_1\beta_2}, & B_{\beta_1\gamma_2} \\ B_{\gamma_1\beta_2}, & B_{\gamma_1\gamma_2} \end{vmatrix} \end{array}\begin{array}{c} \\ (\beta_1) \\ (\gamma_1) \end{array} \qquad Q_{21} = \begin{array}{c} \begin{array}{cc} (\alpha_1) & (\delta_1) \end{array} \\ \begin{vmatrix} Q_{\alpha_2\alpha_1}, & Q_{\alpha_2\delta_1} \\ Q_{\delta_2\alpha_1}, & Q_{\delta_2\delta_1} \end{vmatrix} \end{array}\begin{array}{c} \\ (\alpha_2) \\ (\delta_2) \end{array}$$

$$Q_{C1} = \begin{array}{cc} (\alpha_1) & (\delta_1) \\ {[Q_{C\alpha_1},} & {Q_{C\delta_1}]} \end{array}(C) \qquad Q_{2S} = \begin{array}{c} (S) \\ \begin{vmatrix} Q_{\alpha_2 S} \\ Q_{\delta_2 S} \end{vmatrix} \end{array}\begin{array}{c} \\ (\alpha_2) \\ (\delta_2) \end{array}$$

$$e_1 = \begin{vmatrix} e_{\beta_1} \\ e_{\gamma_1} \end{vmatrix} \qquad j_2 = \begin{vmatrix} j_{\alpha_2} \\ j_{\delta_2} \end{vmatrix}. \tag{5-4.10}$$

The voltage-current relation of capacitors is given by

$$\begin{vmatrix} C_S, & 0 \\ 0 & , C_C \end{vmatrix} \frac{d}{dt} \begin{vmatrix} v_S \\ v_C \end{vmatrix} = \begin{vmatrix} i_S \\ i_C \end{vmatrix}. \tag{5-4.11}$$

The network-equation is given by Eq.(5-4.9) and (5-4.11).

If the matrix B in Eq.(5-4.9) is a zero matrix, it is easy to derive a state equation with the state variables v_C. For passive networks considered in the preceding chapters, the matrix B is always a zero matrix, however, for active networks, it may not be a zero one.

Now let us examine the relations between the matrix B and the network-topology. The matrix B given in Eq.(5-4.10) is rewritten as

$$B=B_{S2}g^{-1}Q_{2S}, \tag{5-4.12}$$

and the matrix g

$$g=[\overset{(\beta_2)+(\gamma_2)}{1}, Q_{21}]\begin{bmatrix} G_2, & 0 \\ 0, & R_1^{-1} \end{bmatrix}\begin{bmatrix} 1 \\ -B_{12} \end{bmatrix}$$

$$\triangleq Q_{ID}GQ'_{VD}. \tag{5-4.13}$$

The first matrix of the right hand side of Eq.(5-4.13) is the fundamental cut-set matrix with respect to $T_c^N(\alpha,\delta)$ for $G_I[T_c^N(C);\bar{T}_c^N(C)\}$ (denoted by G_{ID}). The last matrix of right hand side of Eq.(5-4.13) is the fundamental cut-set matrix with respect to $T_c^N(\beta,\gamma)$ for $G_V[T_c^N(C);\bar{T}_c^N(C)\}$ (denoted by G_{VD}). The second one is diagonal matrix. Then, if the (i,j) element of g^{-1} is nonzero, there exists a common tree in $G_{VD}[twig(q_{VD}(i))]$ and $G_{ID}[twig(q_{ID}(j))]$ since $g^{-1}(i,j)=Cof.g(i,j)/|g|$. Then if a (k,k) element of B is nonzero, there exist $twig(q_{VD}(i))$ in the $loop(link(b_{2S}(k)))$ in G_V, and $link(q_{2S}\{k\})$ in the $cut\text{-}set(twig(q_{VD}(j)))$ in G_I. Therefore there exists a common tree in $G_V[T_c^N(C)+link(b_{S2}(k));$ $\bar{T}_c^N(C)-link(b_{S2}(k))\}$ and $G_I[T_c^N(C)+link(q_{2S}\{k\});\bar{T}_c^N(C)-link(q_{2S}\{k\})\}$. Then the capacitor corresponding to the $link(b_{S2}(k))$ in G_V is identical with that corresponding to the $link(q_{2S}\{k\})$. Thus if every diagonal element of B were nonzero, T_c^N could not be a normal common tree, and we have

Lemma 5-4.1

Every diagonal element of B in Eq.(5-4.9) is zero.

In order to examine properties of B, let us introduce a matrix $B^b=\{b^b(i,j)\}$ defined as follows;

$b^b(i,j)=1$ if $b(i,j)\neq 0$, and $b^b(i,j)=0$ if $b(i,j)=0$. (5-4.14)

The matrix B^b is square and its elements are 0 or 1. Then a digraph can be considered, whose adjacency matrix is B^b. It is denoted by G^b and called the Boolean digraph for B. The nodes in G^b correspond to the link-capacitors. The digraph G^b is a subgraph of $G^\#$ and obtained from $G^\#$ by deleting twig-capacitor-nodes and contracting[@] dependent source-nodes.

Lemma 5-4.2

The directed graph G^b has no directed loop.

Proof: Assume that G^b had a directed loop the length of which is l_d.

From Lemma 5-4.1, there is no loop of length 1.

Assume that $l_d=n$. Let S_1, S_2,$\cdots$ and S_n denote the link-capacitors corresponding to the nodes of the directed loop in G^b. Then corresponding to nonzero (i_k,j_k) elements of g^{-1}, there exist twig-dependent source-branches b_{i_k} in the $loop(S_k)$ of G_V, and twig-dependent source-branches b_{j_k} in the $loop(S_k)$ of G_I.

With loos of generality, it can be assumed that twig-dependent source-branches can be chosen such that b_{i_k} is not identical with b_{i_l}, or b_{j_k} is not identiacl with b_{j_l} $\{k,l=1,\cdots,n.\ k\neq l\}$ since if

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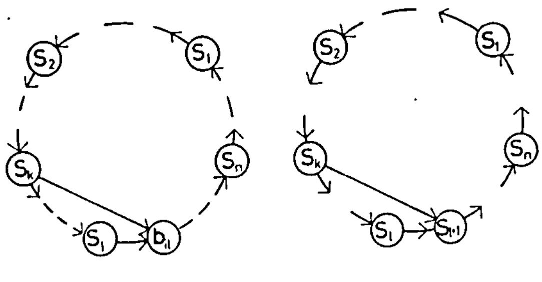
@ Node-contraction is the same as node-elimination in flow graphs.

such branches cannot be chosen, there exists a directed loop of the length less than n. (See Fig.5-4.1) From $g^{-1}(i_k, j_k) \neq 0$, there exists at least one common tree in $G_{VD}[b_{i_1}, \cdots, b_{i_n}]$ and $G_{ID}[b_{j_1}, \cdots, b_{j_n}]$. Then there exists at least one common tree in $G_V[T_c^N(C)+S_1+\cdots+S_n; \bar{T}_c^N(C)-(S_1+\cdots+S_n)]$ and $G_I[T_c^N+S_1+\cdots+S_n; \bar{T}_c^N-(S_1+\cdots+S_n)]$ from Property 5-2.5. However, this contradicts that T_c^N is a normal common tree. Therefore the directed graph G^b has no directed loop.



(a) $G^{\#}$ (b) G^b

Fig.5-4.1 Directed loop in (a) $G^{\#}$ and (b) G^b

Q.E.D.

From Lemma 5-4.2, we have

Lemma 5-4.3

The matrix B^b"B" is a nilpotent matrix.

Proof: Since G^b has no directed loop, there exists a finite number m such that

$$(B^b)^m = 0. \tag{5-4.15}$$

Consequently the matrix B is a nilpotent matrix.

Q.E.D.

From Lemma 5-4.3, the matrix B can be transposed into a strictly triangular matrix $P_1 B P_1'$, where P_1 is a permutation-matrix.

Assume that Eq.(5-4.9) and (5-4.11) can be rewritten as

$$\begin{vmatrix} 1, -F_1, -F_2 \\ 0,\ B_1,\ B_2 \\ 0,\ 0\ ,\ 0 \end{vmatrix} \begin{vmatrix} i_C \\ i_{S1} \\ i_{S2} \end{vmatrix} = \begin{vmatrix} D\ ,0,0 \\ -A_1,1,0 \\ -A_2,0,1 \end{vmatrix} \begin{vmatrix} v_C \\ v_{S1} \\ v_{S2} \end{vmatrix} + \begin{vmatrix} \tilde{e}'_C \\ \tilde{e}'_{S1} \\ \tilde{e}'_{S2} \end{vmatrix} + \begin{vmatrix} \tilde{j}'_C \\ \tilde{j}'_{S1} \\ \tilde{j}'_{S2} \end{vmatrix}, \tag{5-4,16}$$

and

$$\begin{vmatrix} C_C,\ 0\ ,\ 0 \\ 0\ ,C_{S1},\ 0 \\ 0\ ,\ 0\ ,C_{S2} \end{vmatrix} \frac{d}{dt} \begin{vmatrix} v_C \\ v_{S1} \\ v_{S2} \end{vmatrix} = \begin{vmatrix} i_C \\ i_{S1} \\ i_{S2} \end{vmatrix}, \tag{5-4.17}$$

where

$$[F_1, F_2] = FP_1', \quad \begin{vmatrix} B_1, B_2 \\ 0\ ,0 \end{vmatrix} = P_1 B P_1', \quad \begin{vmatrix} A_1 \\ A_2 \end{vmatrix} = P_1 A, \quad \begin{vmatrix} v_{S1} \\ v_{S2} \end{vmatrix} = P_1 v_S$$

$$\begin{vmatrix} \tilde{e}_{S1} \\ \tilde{e}_{S2} \end{vmatrix} = P_1 \tilde{e}_S, \quad \begin{vmatrix} \tilde{j}_{S1} \\ \tilde{j}_{S2} \end{vmatrix} = P_1 \tilde{j}_S, \quad \begin{vmatrix} i_{S1} \\ i_{S2} \end{vmatrix} = P_1 i_S, \quad \begin{vmatrix} C_{S1}, & 0 \\ 0 & ,C_{S2} \end{vmatrix} = P_1 C_S P_1'$$

The matrix B_1 is strictly triangular.

The link-capacitors S_2 are such that there exist no paths from the nodes S_2 to other link-capacitor-nodes in G^b, that is, no paths not containing twig-capacitor-nodes in $G^{\#}$.

Eliminating v_{S2} and i_{S2} from Eq.(5-4.16), we obtain

$$\left|\begin{array}{c|c} 1-F_2 C_{S2} A_2 C_C^{-1}, & -F_1 \\ \hline B_2 C_{S2} A_2 C_C^{-1}\ , & B_1 \end{array}\right| \begin{vmatrix} i_C \\ i_{S1} \end{vmatrix} = \left|\begin{array}{c|c} D\ , & 0 \\ \hline -A_1, & 1 \end{array}\right| \begin{vmatrix} v_C \\ v_{S1} \end{vmatrix} + \begin{vmatrix} \tilde{e}_C \\ {}_1\tilde{e}_{S1} \end{vmatrix} + \begin{vmatrix} \tilde{j}_C \\ {}_1\tilde{j}_{S1} \end{vmatrix}$$
$$+ \begin{vmatrix} -F_2 \\ B_2 \end{vmatrix} C_{S2}(\dot{\tilde{e}}_{S2} + \dot{\tilde{j}}_{S2}). \tag{5-4.18}$$

Multiplying both sides of Eq.(5-4.18) on the left by the non-singular matrix,

$$\left|\begin{array}{c|c} 1 & ,\ 0 \\ \hline -B_2 C_{S2} A_2 (C_C - F_2 C_{S2} A_2)^{-1} & ,\ 1 \end{array}\right| \tag{5-4.19}$$

we obtain

$$\begin{vmatrix} {}_1E_1, & -F_1 \\ 0, & B_1+\widetilde{B}_1 \end{vmatrix} \begin{vmatrix} i_C \\ i_{S1} \end{vmatrix} = \begin{vmatrix} D, & 0 \\ -{}_1A, & 1 \end{vmatrix} \begin{vmatrix} v_C \\ v_{S1} \end{vmatrix} + \begin{vmatrix} \widetilde{e}_C \\ {}_1\widetilde{e}_{S1} \end{vmatrix} + \begin{vmatrix} \widetilde{j}_C \\ {}_1\widetilde{j}_{S1} \end{vmatrix} + \begin{vmatrix} -F_2 \\ {}_1B_2 \end{vmatrix} C_{S2}(\dot{\widetilde{e}}_{S2}+\dot{\widetilde{j}}_{S2}) \tag{5-4.20}$$

where

$${}_1E_1 = 1-F_2C_{S2}A_2C_C^{-1},$$
$$\widetilde{B}_1 = B_2C_{S2}A_2(C_C-F_2C_{S2}A_2)^{-1}F_1$$
$${}_1A = A_1+B_2C_{S2}A_2(C_C-F_2C_{S2}A_2)^{-1}D$$
$${}_1\widetilde{e}_{S1} = -B_2C_{S2}A_2(C_C-F_2C_{S2}A_2)^{-1}\widetilde{e}_C+\widetilde{e}_{S1}$$
$${}_1\widetilde{j}_{S1} = -B_2C_{S2}A_2(C_C-F_2C_{S2}A_2)^{-1}\widetilde{j}_C+\widetilde{j}_{S1}$$
$${}_1B_2 = B_2C_{S2}A_2(C_C-F_2C_{S2}A_2)^{-1}F_2+B_2.$$

Let us examine properties of the matrix ${}_1B$ defined as :

$${}_1\widetilde{B}=B_1+\widetilde{B}_1. \tag{5-4.21}$$

Lemma 5-4.4

If there exist nonzero elements in ${}_1\widetilde{B}$, then the induced digraph $G^{\#}$ has subgraphs such as shown in Fig.5-4.2, where S."C." denote link-"twig-"capacitors.

Proof: As described in the proofs in Lemma 5-4.1 - 5-4.3, the graph shown in Fig.5-4.2(a) is obtained for nonzero elements of B_1.

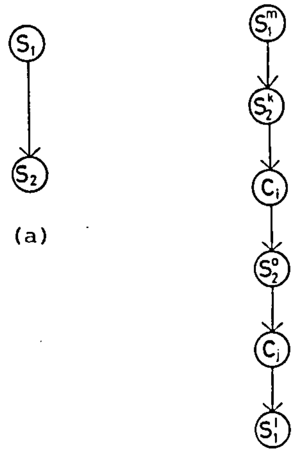
Assume that there exists a nonzero (m,l) element of the matrix B_1. Since B_1 is written as

$$\widetilde{B}_1=B_2C_{S2}A_2\widetilde{C}_C^{-1}F_1, \text{ (where } \widetilde{C}_C=C_C-F_2C_{S2}A_2) \tag{5-4.22}$$

there exist k, i and j such that

$$b_2(m,k)C_{S2}(k,k)a_2(k,i)\widetilde{C}_c^{-1}(i,j)f_1(j,l)\neq 0. \tag{5-4.23}$$

Let S_1^m and S_2^k denote the link-capacitors corresponding to $b_2(m)$ and $b_2\{k\}$, then the induced digraph $G^{\#}$ should include a sub-



(a)

(b)

Fig.5-4.2 Subgraphs in $G^{\#}$ for Lemma 5-4.4

Fig.5-4.3 Subgraph corresponding to nonzero element of B_2.

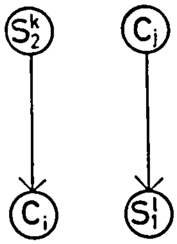


Fig.5-4.4 Subgraphs corresponding to $a_2(k,j)\neq 0$ and $f_1(j,l)\neq 0$.

graph shown in Fig.5-4.3 because $b_2(m,k)\neq 0$.

Let C_i and C_j denote the twig-capacitors corresponding to $a_2\{i\}$ and $f_1(j)$, respectively, and S_1^l, the link-capacitor to $f_1\{l\}$. Then $G^{\#}$ should include a subgraph shown in Fig.5-4.4 because $a_2(k,i)\neq 0$ and $f_1(j,l)\neq 0$.

Consider $\tilde{C}_C^{-1}(i,j)$ defined in Eq.(5-4.22). The matrix C_C can be rewritten as

$$\tilde{C}_C=[1,F_2]\begin{vmatrix} C_C, & 0 \\ 0 & ,C_{S2} \end{vmatrix}\begin{vmatrix} 1 \\ -A_2 \end{vmatrix}. \qquad (5\text{-}4.24)$$

Similarly to that with respect to the relation between $g^{-1}(i,j)\neq 0$ in Eq.(5-4.13) and the network-topology, the following argument holds. Since $\tilde{C}_C^{-1}(i,j)\neq 0$, there exists a link-capacitor S_2^o and a common tree in $G_V[C_i;\bar{T}_c^N(C)-S_2^o\}$ and $G_I[C_i;\bar{T}_c^N(C)-S_2^o\}$. The common tree contains $T_c^N(C)-C_i-C_j$ and S_2^o. Therefore if

$C_C^{-1}(i,j) \neq 0$, there exists a subgraph in $G^{\#}$ such as shown in Fig.5-4.5. From Fig.5-4.3 - 5-4.5, we see that $G^{\#}$ includes graphs in Fig.5-4.2 if there exists a nonzero element of ${}_1B$.

Q.E.D.

Fig.5-4.5 Subgraph corresponding to nonzero element of C_C^{-1}.

Lemma 5-4.5

The graph ${}_1G^b$ has no directed loop and the ${}_1\tilde{B}$ is a nilpotent matrix, where ${}_1G^b$ is the Boolean digraph for ${}_1\tilde{B}$.

Proof: If there is no sequence $i_0, i_1, \cdots$ and $i_n \{n=1,\cdots\}$ such that all (i_0,i_1), $(i_1,i_2),\cdots,(i_{n-1},i_n)$ and (i_n,i_0) elements of ${}_1B$ are nonzero, ${}_1G^b$ has no directed loop, and then the matrix ${}_1\tilde{B}$ is a nilpotent matrix.

Assume that all (i_0,i_1), $(i_1,i_2),\cdots,(i_{n-1},i_n)$ and (i_n,i_0) elements of ${}_1\tilde{B}$ are nonzero. Then, from Lemma 5-4.4, the induced digraph $G^{\#}$ includes a directed loop D_L such as is shown in Fig.5-4.6, for $n=1$, as an example. No capacitor-node in the directed loop is not identical with another.

In order to examine the directed loop D_L, let us consider the graphs G_V' and G_I' obtained from G_V and G_I by contracting all twigs and deleting all links except the capacitor- and the dependent source-

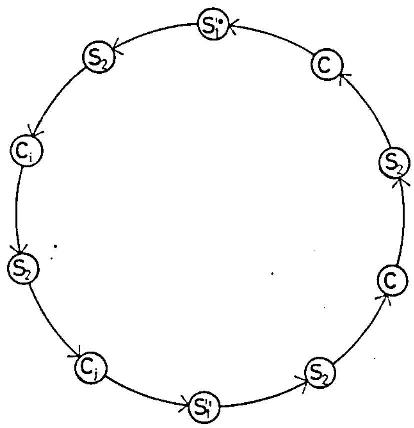


Fig.5-4.6 Example of directed loop D_L.

branches corresponding to the nodes in D_L. There exists a subset of T_c^N (denoted by $T_c^{N\prime}$) such that $T_c^{N\prime}$ is a common tree of G_V' and G_I'. If the common cotree $\bar{T}_c^{N\prime}$ for $T_c^{N\prime}$ were a common tree, T_c^N would not be a normal common tree since $\bar{T}_c^{N\prime}$ contains more capacitor-branches than $T_c^{N\prime}$.(See Fig.5-4.7). Then all (i_0,i_1), (i_1,i_2), $\cdots$, (i_n,i_0) elements of ${}_1\tilde{B}$ are not nonzero.

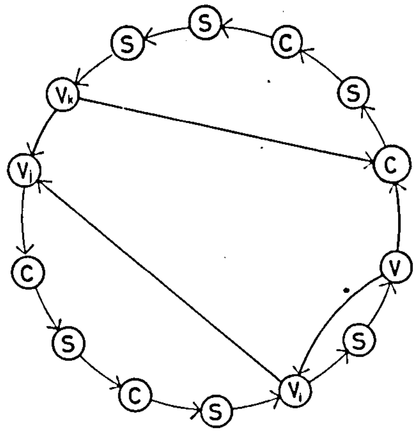


Fig.5=4.7
Directed loop and subloop.

If $\bar{T}_c^{N\prime}$ is not a common tree, there exist some directed subloops in $(G^\#)'$, where $(G^\#)'$ is the induced graph corresponding to G_V' and G_I', from Property 5-2.4 in Section 5-2. Moreover there exists a directed subloop in $(G^\#)'$ such that the number of the link-capacitor-nodes in the subloop is greater than that of the twig-capacitor-nodes. This is because if there exists a directed subloop where the number of the twig-capacitor-nodes is greater than (or equal to) that of the link-capacitor-nodes, for example, such as the directed sub-loop $\{V_j,\cdots,C,\cdots,S,\cdots,C,\cdots,V_i,V_j\}$ shown in Fig.5-4.7, where V_i is a capacitor- or dependent source-node, then from Property 5-2.4, there exists at least one directed loop D_L' such that the number of the link-capacitor-nodes is greater than that of the twig-capacitor-nodes.(See Fig.5-4.7). Consider the graphs G_V'' and G_I'' obtained from G_V' and G_I' by contracting all twigs and deleting all links except the capacitor- and dependent source-branches corresponding to the nodes in the directed subloop D_L'. Let $T_c^{N\prime\prime}$

dencte a subset of $T_c^{N\prime}$, where $T_c^{N\prime\prime}$ is a common tree of G_V'' and G_I''. If $\bar{T}_c^{N\prime\prime}$ were a common tree of G_V'' and G_I'', T_c^N would not be a normal common tree. If $\bar{T}_c^{N\prime\prime}$ is not a common tree, there exists a directed subloop D_L'' in $(G^{\#})''$, where the number of the link-capacitor-nodes is greater than that of the twig-capacitor-nodes.

By repeating the above discussion, we can conclude that if all (i_0,i_1), $(i_1,i_2),\cdots$, (i_{n-1},i_n) and (i_n,i_0) elements of ${}_1\widetilde{\mathrm{B}}$ are nonzero, there exists a directed loop $D_L^{(k)}$ and $(G^{\#})^{(k)}$, such that the number of the link-capacitor-nodes is greater than that of the twig-capacitor-nodes. From Property 5-2.5, it can be proved that $\bar{T}_c^{N(k)}$ is a common tree of $G_V^{(k)}$ and $G_I^{(k)}$ obtained from G_V and G_I by contracting all twigs and deleting all links except the branches corresponding to the nodes in the directed loop $D_L^{(k)}$. Then, the common tree $T_c^N(b)+\bar{T}_c^{N(k)}$, where b is the set of tiwgs except those in $D_L^{(k)}$, would contain more capacitor-branches than T_c^N. This contradicts that T_c^N is a normal common tree. Therefore there is no sequence i_0, $i_1,\cdots$ and i_n such that all $(i_0,i_1),\cdots$, (i_{n-1},i_n) and (i_n,i_0) elements of ${}_1\widetilde{\mathrm{B}}$ are nonzero, and ${}_1G^b$ has no directed loop. Therefore the matrix ${}_1\widetilde{\mathrm{B}}$ is a nilpotent matrix.

Q.E.D.

From Lemma 5-4.5, by a permutation matrix P_2, we obtain a strictly triangular matrix,

$$\mathrm{P}_2\,{}_1\widetilde{\mathrm{B}}\mathrm{P}_2' = \begin{vmatrix} {}_1\widetilde{\mathrm{B}}_1, & {}_1\widetilde{\mathrm{B}}_2 \\ 0, & 0 \end{vmatrix}. \qquad (5\text{-}4.25)$$

By use of the matrix P_2, Eq.(5-4.20) becomes

$$\begin{vmatrix} {}_1E_1, -{}_1F_1, -{}_1F_2 \\ 0\ ,\ {}_1\widetilde{B}_1,\ {}_1\widetilde{B}_2 \\ 0\ ,\ 0\ ,\ 0 \end{vmatrix} \begin{vmatrix} i_C \\ {}_2i_{S1} \\ {}_2i_{S2} \end{vmatrix} = \begin{vmatrix} D\ ,0,0 \\ -{}_1\widetilde{A}_1,1,0 \\ -{}_1\widetilde{A}_2,0,1 \end{vmatrix} \begin{vmatrix} v_C \\ {}_2v_{S1} \\ {}_2v_{S2} \end{vmatrix} + \begin{vmatrix} \widetilde{e}_C \\ {}_2\widetilde{e}_{S1} \\ {}_2\widetilde{e}_{S2} \end{vmatrix} + \begin{vmatrix} \widetilde{j}_C \\ {}_2\widetilde{j}_{S1} \\ {}_2\widetilde{j}_{S2} \end{vmatrix}$$

$$+ \begin{vmatrix} -F_3 \\ {}_1\widetilde{B}_{21} \\ {}_1\widetilde{B}_{31} \end{vmatrix} C_S[\dot{\widetilde{e}}_{S3} + \dot{\widetilde{j}}_{S3}], \qquad (5\text{-}4.26)$$

where

$$P_2 i_{S1} = \begin{vmatrix} {}_2i_{S1} \\ {}_2i_{S2} \end{vmatrix}, \qquad P_2 v_{S1} = \begin{vmatrix} {}_2v_{S1} \\ {}_2v_{S2} \end{vmatrix}, \qquad P_{21}A_1 = \begin{vmatrix} {}_1\widetilde{A}_1 \\ {}_1\widetilde{A}_2 \end{vmatrix}$$

$$P_{21}\widetilde{e}_{S1} = \begin{vmatrix} {}_2\widetilde{e}_{S1} \\ {}_2\widetilde{e}_{S2} \end{vmatrix}, \qquad P_{21}\widetilde{j}_{S1} = \begin{vmatrix} {}_2\widetilde{j}_{S1} \\ {}_2\widetilde{j}_{S2} \end{vmatrix}, \qquad P_{21}B_2 = \begin{vmatrix} {}_1\widetilde{B}_{21} \\ {}_1\widetilde{B}_{31} \end{vmatrix}$$

$$-F_1P_2' = [-{}_1F_1, -{}_1F_2].$$

The vectors ${}_2i_{S2}$ and ${}_2v_{S2}$ can be eliminated from Eq.(5-4.26). Then the following Eq.(5-4.27), which is of similar form to Eq.(5-4.20), may be obtained.

$$\begin{vmatrix} {}_2E, -{}_2F \\ 0,\ {}_2\widetilde{B} \end{vmatrix} \begin{vmatrix} i_C \\ {}_2i_{S1} \end{vmatrix} = \begin{vmatrix} D,0 \\ -{}_2A,1 \end{vmatrix} \begin{vmatrix} v_C \\ {}_2v_{S1} \end{vmatrix} + \begin{vmatrix} \widetilde{e}_C \\ {}_1\widetilde{e}_{S1,0} \end{vmatrix} + \begin{vmatrix} \widetilde{j}_C \\ {}_1\widetilde{j}_{S1,0} \end{vmatrix} + \begin{vmatrix} \dot{\widetilde{e}}_{C1,1} \\ {}_1\dot{\widetilde{e}}_{S1,1} \end{vmatrix}$$

$$+ \begin{vmatrix} \dot{\widetilde{j}}_{C1,1} \\ {}_1\dot{\widetilde{j}}_{S1,1} \end{vmatrix} + \begin{vmatrix} \ddot{\widetilde{e}}_{C1,2} \\ {}_1\ddot{\widetilde{e}}_{S1,2} \end{vmatrix} + \begin{vmatrix} \ddot{\widetilde{j}}_{C1,2} \\ {}_1\ddot{\widetilde{j}}_{S1,2} \end{vmatrix}. \qquad (5\text{-}4.27)$$

A subgraph of $G^{\#}$ which corresponds to the nonzero element of ${}_2\widetilde{B}$ has the same property as that of ${}_1\widetilde{B}$. Then it can be proved in the same manner as the proof in Lemma 5-4.5 that ${}_2G^b$ has no directed loop and ${}_2\widetilde{B}$ is a nilpotent matrix. Therefore there exists a permutation matrix P_3 such that $P_{3\,2}\widetilde{B}P_3'$ is strictly triangular. The vector ${}_2v_{S1}$ and ${}_2i_{S1}$ are partitioned into

$$P_{32}i_{S1}=\begin{vmatrix} {}_3i_{S1} \\ {}_3i_{S2} \end{vmatrix}, \qquad P_{32}v_{S1}=\begin{vmatrix} {}_3v_{S1} \\ {}_3v_{S2} \end{vmatrix}. \tag{5-4.28}$$

Then the vectors ${}_3v_{S2}$ and ${}_3i_{S2}$ can be eliminated from Eq.(5-4.27), and an equation of similar form to Eq.(5-4.27) can be obtained.

Repeating the above procedure, we obtain

$$\begin{vmatrix} {}_nE, -{}_nF \\ 0, {}_n\widetilde{B} \end{vmatrix} \begin{vmatrix} i_C \\ {}_ni_S \end{vmatrix} = \begin{vmatrix} D, 0 \\ -{}_nA, 1 \end{vmatrix} \begin{vmatrix} v_C \\ {}_nv_S \end{vmatrix} + \Sigma\left\{ \begin{vmatrix} {}_n e_C^{(n)} \\ {}_n e_S^{(n)} \end{vmatrix} + \begin{vmatrix} {}_n J_C^{(n)} \\ {}_n J_S^{(n)} \end{vmatrix} \right\}, \tag{5-4.29}$$

where ${}_0E=1$, ${}_0F=F$, ${}_0\widetilde{B}=B$, ${}_0A=A$.

Now let us consider the properties of the induced directed graph when there are nonzero elements in the matrices ${}_nA$, ${}_n\widetilde{B}$, ${}_nF$ and D.

Property 5-4.1

If there are nonzero elements in A, B, F and D in Eq.(5-4.9), there exist directed paths in $G^{\#}$, which are denoted by P_A, P_B, P_F and P_D, respectively, as shown in Fig.5-4.8(a).

Proof: The matrix A is written by

$$A=B_{SC}+B_{S2}\,g^{-1}Q_{21}R_1^{-1}B_{1C}.$$

Then the nonzero elements of A correspond to directed paths, as shown in Fig.5-4.8(a), whose initial nodes are link-capacitor-nodes and terminal nodes are twig-capacitor-nodes. Similarly nonzero elements of B, F and D correspond to directed paths as shown in Fig.5-4.8(a). The initial"terminal" nodes of P_A, P_F and P_D are link-"link-", twig-"link-" and twig-"twig-" capacitor nodes, respectively.

Q.E.D.

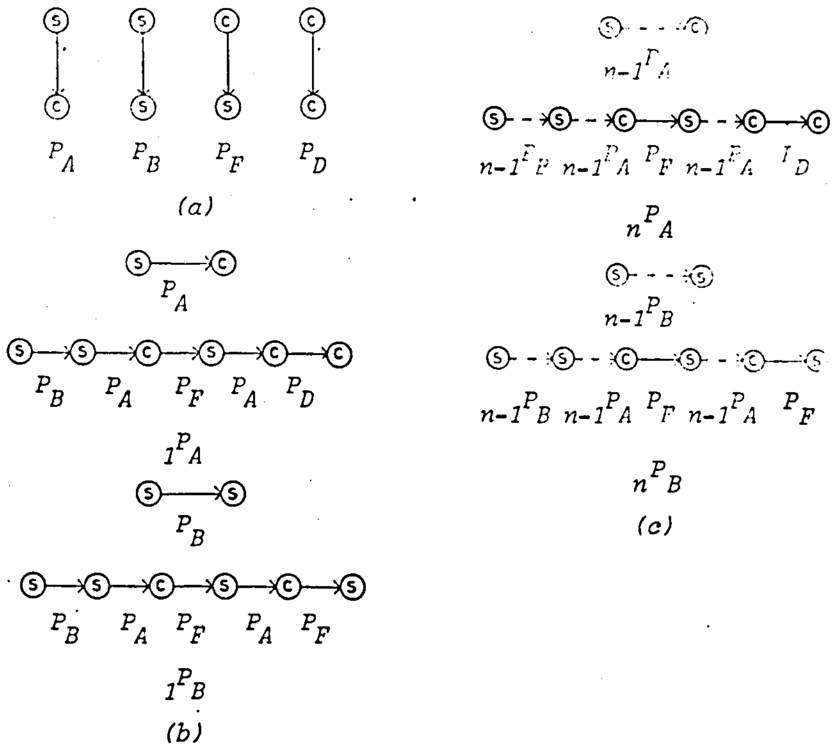


Fig.5-4.8 Paths in $G^{\#}$ corresponding to nonzero elements of the matrices.

Property 5-4.2

If there exists a nonzero element in ${}_nA$ and ${}_n\tilde{B}$ $\{n=1,2,\cdots\}$, there exists a directed path as shown in Fig.5-4.8(b) for $n=1$ and Fig.5-4.8(c) for $n\geq 2$. The paths are denoted by ${}_nP_A$ and ${}_nP_B$, respectively.

Proof: Examine the case of $n=1$. From Eq.(5-4.20), we have

$${}_1A = A_1 + B_2 C_{S2} A_2 (C_C - F_2 C_{S2} A_2)^{-1} D$$

$${}_1B = B_1 + B_2 C_{S2} A_2 (C_C - F_2 C_{S2} A_2)^{-1} F_2$$

where the matrices B_1 and B_2 are submatrices of B. The matrix ${}_nF$ is a submatrix of F. The matrix D is invariant during the elimination-procedures.

A nonzero element of $(C_C-F_2C_{S2}A_2)^{-1}$ (cf. Lemma 5-4.4) corresponds to a path, which has twig-capacitor-nodes as its initial and terminal nodes and contains a link-capacitor-node. The path is represented by P_F+P_A. Then the path corresponding to a nonzero element of ${}_1A$ is represented by P_A or $P_B+P_A+P_F+P_A+P_D$, as is shown in Fig.5-4.8(b). Similarly the path ${}_1P_B$ is represented by P_B or $P_B+P_A+P_F+P_A+P_F$, which is shown in Fig.5-4.8(b).

By induction, the path ${}_nP_A$ is represented by ${}_{n-1}P_A$ or ${}_{n-1}P_B+{}_{n-1}P_A+P_F+{}_{n-1}P_A+P_D$. Similarly, the path ${}_nP_B$ is represented by ${}_{n-1}P_B$ or ${}_{n-1}P_B+{}_{n-1}P_A+P_F+{}_{n-1}P_A+P_F$.

Q.E.D.

Let $N_S(.P.)$"$N_C(.P.)$" denote the number of link-"twig-" capacitor-nodes in $.P.$. The number $N(.P.)$ is defined by

$N(.P.)\triangleq N_S(.P.)-N_C(.P.)$.

Property 5-4.3

The following equalities hold:

$N({}_nP_A)=0,\quad N({}_nP_B)=2,\quad N(P_F)=0,\quad N(P_D)=-2.$

Proof: From Property 5-4.1, the following equalities hold:

$N(P_A)=0,\quad N(P_B)=2,\quad N(P_F)=0,\quad N(P_D)=-2.$

From Property 5-4.2, we obtain

$N({}_nP_A)=0,\quad N({}_nP_B)=2.$

Q.E.D.

From the above properties, we obtain

Lemma 5-4.6

The matrix ${}_n\tilde{B}$ is a nilpotent matrix.

Proof: If ${}_n\tilde{B}$ were not a nilpotent matrix, there would exist a directed loop in $G^{\#}$, which consists of some directed paths ${}_nP_B$. There would exist more link-capacitor-nodes than twig-capacitor-nodes in the directed loop. Then in the similar manner to the proof of Lemma 5-4.5, it can be proved that the matrix ${}_n\tilde{B}$ is a nilpotent matrix.

Q.E.D.

From Lemma 5-4.6, repeating the elimination-procedure, we can obtain an equation of similar form to Eq.(5-4.29), where ${}_m\tilde{B}=0$. Therefore a state equation can be obtained whose state variables are the voltages across the twig-capacitors, that is,

$$\dot{v}_C = A v_C + B_0 u + \cdots\cdots + B_{m+1} \overset{(m+1)}{u}. \qquad (5\text{-}4.30)$$

Note that the matices A and B_i in this equation are different from those in Eq.(5-4.27).

Then we have

Theorem 5-4.1

An active network on the Assumption 5-4.1 which contains capacitors, voltage-controlled current-sources, current-controlled voltage-sources and independent sources has a state equation whose state variables are the voltages across the twig-capacitors in a normal common tree.

The state equation for a passive network may have first order-derivatives of inputs, $\dot{u}$, but cannot have higher order-derivatives than the first unless there are special relations among the network-element-values. However, the state equation of an active network may have the higher order-derivatives of inputs, $\overset{(n)}{u}$, even if there is no special relation among the network-element-values.

Let NE_0 represent Eq. (5-4.9) and (5-4.11). The equations obtained from NE_0 by the elimination of voltages across and currents through the link-capacitors not corresponding to the initial nodes of ${}_1P_B$, are represented by NE_1. (cf. Eq. (5-4.26)). The equation NE_n represents Eq. (5-4.29) plus the voltage- and current-relations of capacitors.

If there exists a link-capacitor S_k $\{k \geq 1\}$ whose voltage and current are contained in N_k but not in N_{k-1}, then there exixts a link-capacitor S_{k-1} such that 1) there exists a path ${}_{k-1}P_B$ from S_k to S_{k-1} in $G^{\#}$, 2) its voltage and current are contained in NE_{k-1} but not in NE_k, and 3) S_{k-1} does not correspond to the initial nodes of ${}_{k-1}P_B$. Similarly for S_{k-1}, there exists a link-capacitor S_{k-2} such that 1) there exists a path ${}_{k-2}P_B$ from S_{k-1} to S_{k-2} in $G^{\#}$, 2) Its voltage and current are contained in NE_{k-2} but not in NE_{k-1}, and 3) S_{k-2} does not correspond to the initial nodes of ${}_{k-2}P_B$, and so on. Consequently for S_k, there exists a link-capacitor S_0 such that 1) there exists a path P (represented by $P={}_{k-1}P_B+\cdots+P_B$) from S_k to S_0 in $G^{\#}$, 2) its voltage and current are contained in NE_0 but not in NE_1, and 3) S_0 does not correspond to the initial nodes of every ${}_lP_B$ $\{l=1,2,\cdots\}$. For the path P, $N(P)=k+1$.

Definition 5-4.1

A directed path in $G^{\#}$ whose initial and terminal nodes link-capacitor-nodes is called an l-path and denoted by L_d. A maximum l-path is an l-path such that $N(L_d)$ is maximum. It is denoted by L_{md}. The number l_m is defined by $l_m \triangleq N(L_{md})$.

If a state equation is obtained from NE_0 by l_m-repetitions of the eliminating procedure of the voltages across and the currents through link-capacitors, then there exists an l-path L_d with $N(L_d)=l_m$, and the state equation may have l_m-th order-derivatives of the inputs. Then we obtain

Theorem 5-4.2

In deriving a state equation from the network equations (Eq.(5-4.9) and (5-4.11)), we need not repeat more than l_m times of the eliminating procedure. The order of derivatives of inputs is not more than l_m.

For example, consider the derivation of a state equation for the network whose induced digraph has the subgraph shown in Fig.5-4.9. In this digraph, the directed paths P_A, P_B, P_F and P_D are

P_A: S_3-C_1, S_4-C_2

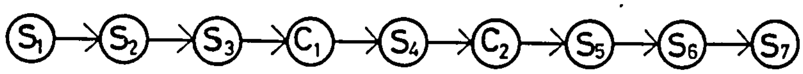


Fig.5-4.9 Example of subgraph of $G^{\#}$.

P_B: S_1-S_2, S_2-S_3, S_5-S_6, S_6-S_7

P_F: C_1-S_4, C_2-S_5

P_D: ϕ.

Then the equation NE_1 does not contain the voltages across and the currents through the link-capacitors S_3, S_4 and S_7, which do not correspond to the initial nodes of P_B. The matrix ${}_1\tilde{B}$ may be

$$\begin{array}{c} \\ S_1 \\ S_2 \\ S_5 \\ S_6 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_5 & S_6 \end{array} \\ \left|\begin{array}{cccc} & *1 & & \\ & & *2 & \\ & & & *3 \\ & & & \end{array}\right| \end{array} .$$

(where the marks $*i$ mean nonzero. $\{i=1,2,3\}$)

where ${}_1P_B$ are S_1-S_2, S_2-S_3-C_1-S_4-C_2-S_5 and S_5-S_6 for $*1$, $*2$ and $*3$, respectively. Then the equation NE_2 does not contain the voltage across and the current through the link-capacitor S_6. (The link-capacitors S_2 and S_5 correspond to the initial nodes of ${}_1P_B$.)

By induction, the equation NE_3 does not contain the voltage across and the current through the link-capacitor S_5. Consequently the equation NE_4 does not contain the voltages and the currents through all link-capacitors. Then the state equation may have at most sixth order-derivatives of inputs. The number l_m is five in the subgraph shown in Fig.5-4.9.

Consider the matrices appearing in the formulation of the state equation. Nonsingular R and g guarantee the existence of a hybrid matrix, whose rows correspond to the currents through the twig-capacitors and the voltages across the link-capacitors,

and whose columns correspond to the voltages across the twig-capacitors and the currents through the link-capacitors.
The matrices R and g correspond to the loop resistance matrix and the cut-set conductance matrix, respectively, in RLC networks.
The matrix $B(\triangleq B_{S2}g^{-1}Q_{2S})$ may be regarded as a resistance matrix of a non-reactive network, whose ports are connected to the link-capacitors. Nonsingular $\widetilde{C}_C(\triangleq C_C-F_2C_{S2}A_2)$ guarantees the existence of a state equation for the network when the link-capacitors except S_2 are deleted from the network.

5-5 A CANONICAL FORM OF NETWORK-EQUATIONS WITH NO UNIQUE SOLUTION

A state equation of a network with a unique solution is obtained in Section 5-4. It is a canonical form of the network-equation with a unique solution. In this section, is studied a cannonical form of a network-equation with no unique solution. (that is, there exists no common tree.) The matrix pencil[27] is avaiable for the study. (See Appendix I and II.) A matrix pencil is represented by

$$A+\lambda B, \tag{5-5.1}$$

where A and B are $m\times n$ constant matrices and λ is a parameter. In Appendix I, it is shown that pencil $A+\lambda B$ is strictly equivalent to a canonical quasi-diagonal form, $A+\lambda B$. In Appendix II, it is shown that in accordance with the canonical form, a system of m linear differental equations of the first order in n unknown functions with constant coefficients splits into the five cases of subsystems. The subsystems belonging to Case 1) are not consistent with no special relation among the independent source-values, those to Case 2) are always consistent but indeterminate, those to Case 3) are not consistent with no special relation of the derivative forms among the independent source-values, those to Case 4) may have the derivative parts of the source-values, and those to Case 5) posses the unique solution.

The coefficient matrix $\theta(p)$ in Eq.(5-3.1) is written as

$$\theta(p)=\theta_1+p\theta_2, \tag{5-5.2}$$

where

$$\boldsymbol{\theta}_1=\begin{vmatrix} 0 & , 0 & , -1 & , 0 \\ 0 & , H_{VD} & , 0 & , -H_{ID} \\ B_{VC} & , B_{VD} & , 0 & , 0 \\ 0 & , 0 & , Q_{IC} & , Q_{ID} \end{vmatrix}$$

$$\boldsymbol{\theta}_2=diag.\{C,0,0,0\}.$$

Then $\boldsymbol{\theta}(p)$ is a pencil.

Let us classify branches in accordance with a normal maximum tree pair, as shown in Table 5-5.1.

The branches belonging to Type 1 are links in G_V and G_I, those to Type 2 are twigs in G_V and G_I, those to Type 3 are links in G_V but twigs in G_I, and those to Type 4 are twigs in G_V but links in G_I.

Type	G_V	G_I
1	link	link
2	twig	twig
3	link	twig
4	twig	link

Table 5-5.1
Branch-classification in accordance with a normal maximum tree pair.

Since the network cosidered here has no common tree, there exists at least one branch belonging to Type 3 or 4. Then the canonical quasi-diagonal form of the pencil (5-5.2) should contain quasi-diagonal pencil 0, L or L'.[@]

Let us assume that there exist branches of Type 3, and examine the column rank of the matrix M_ε[@]. The matrix M_ε $\{\varepsilon=0\}$ is written as

$$M = \begin{vmatrix} \theta_1 \\ \theta_2 \end{vmatrix} = \begin{array}{cc} \begin{array}{cccc} (n_C) & (n_D) & (n_C) & (n_D) \end{array} & \\ \left| \begin{array}{cccc} 0, & 0, & -1, & 0 \\ 0, & H_{VD}, & 0, & -H_{ID} \\ B_{VC}, & B_{VD}, & 0, & 0 \\ 0, & 0, & Q_{IC}, & Q_{ID} \\ C, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{array} \right| & \begin{array}{l} (n_C) \\ (n_D) \\ (\mu(G_V)) \\ (r(G_I)) \\ (n_C) \\ (n_D) \\ (\mu(G_V)) \\ (r(G_I)) \end{array} \end{array} \qquad (5\text{-}5.3)$$

i) Assume that there is no dependent source-branch belonging to Type 4. Let us choose pivot-terms of a minor determinant of M_0 as follows:

1) In $[B_{VC}, B_{VD}]$, elements corresponding to the branches belonging to Type 1 and 3. (n_1+n_3 elements)

2) In $[Q_{IC}, Q_{ID}]$, elements to Type 2 and 3. (n_2+n_3 elements)

3) In H_{VD}, elements to Type 2. (n_{2D} elements)

4) In $-H_{ID}$, elements to Type 1. (n_{1D} elements)

5) In -1, elements to Type 1. (n_{1C} elements)

6) In C, elements to Type 2 and 4. ($n_{2C}+n_{4C}$ elements)

The independent $2n_e$ pivots can be chosen in the above procedure. Then the rank of M_0 is $2n_e$. If the pivot-terms of a minor

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@ See Appendix I.

determinant of M_ε $\{\varepsilon=1,2,\cdots\}$ are chosen as above, the rank of M_ε is $2(\varepsilon+1)n_e$. The canonical quasi-diagonal form of the coefficient matrix $\theta(p)$ contains no submatrix L_ε, but may contain submatrices L'_η.

ii) Consider the case where there are branches belonging to Type 4

Definition 5-5.1

A Type 2 capacitor-branch in *loop(b)* in G_I (where *b* is a dependent source-branch belonging to Type 4) is called an e-capacitor-branch.

Let us choose pivot-terms of a minor determinant of M_0 as follows:

1) In $[B_{VC}, B_{VD}]$, elements corresponding to the branches belonging to Type 1 and 3.

2) In C, elements to Type 2 and 4.

3) In H_{VD}, elements to Type 2 and 4.

4) In H_{ID}, elements to Type 1.

5) In Q_{ID}, elements to Type 2 and 3.

6) In Q_{ID}, elements corresponding to Type 4 dependent source-branches if the fundamental loops determined by those dependent source-branches contain e-capacitor-branches.

7) In Q_{IC}, elements to Type 2 and 3 except e-capacitor-branches.

8) In 1, elements corresponding to the capacitor-branches not chosen in 7).

If there is not an e-capacitor-branches for every Type 4 dependent source-branch, then the rank of M_0 is less than $2n_e$.

From i) and ii), we have

Lemma 5-5.1

If there is no branch of Type 4, the rank of M_ε $\{\varepsilon=0,1,\cdots\}$ is $2(\varepsilon+1)n_e$.

If there exists an e-capacitor-branch for every dependent source-branch, the rank of M_0 is $2n_e$. Otherwise it is less than $2n_e$.

In Fig.5-5.1, the shaded portions are independent pivots, and

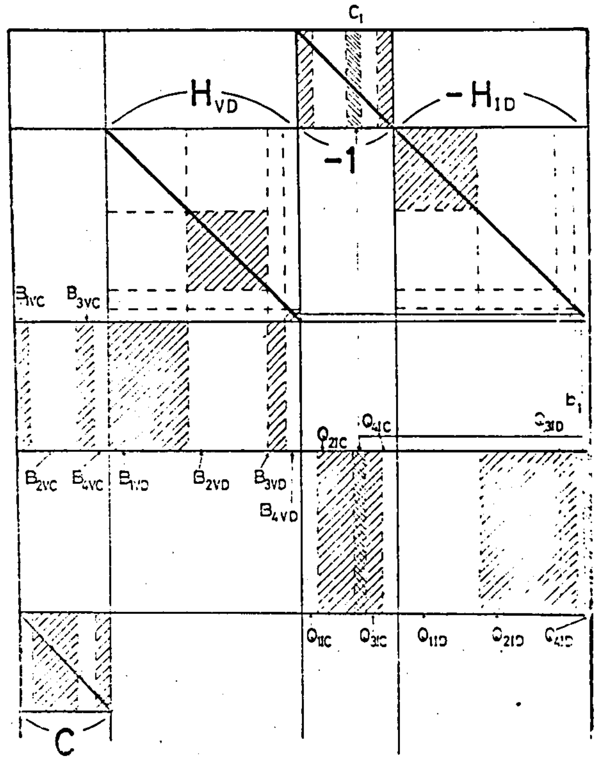


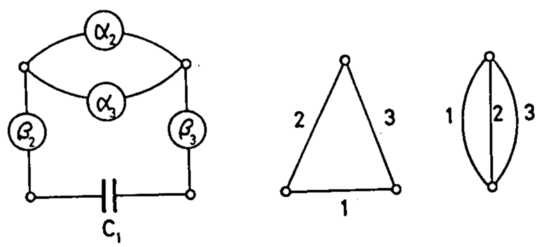
Fig.5-5.1 M_0 and independent pivots.

C_1 are e-capacitor-branches for b_1.

The matrix M_0, for example, for the network shown in Fig.5-5.2, is written as

$$M_0=\begin{vmatrix} 0, & 0, & 0, & -1, & 0, & 0 \\ 0, & g_2, & 0, & 0, & -1, & 0 \\ 0, & 0, & g_3, & 0, & 0, & -1 \\ 1, & 1, & 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ C_1, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \end{vmatrix}. \qquad (5\text{-}5.4)$$

The rank of M_0 is $2n_e=6$. The capacitor C_1 is an e-capacitor for the Type 4 branch 2 in the normal maximum tree-pair $\{T_V^N : T_I^N\}$, where $T_V^N=\{C_1, 2\}$ and $T_I^N=\{C_1\}$.

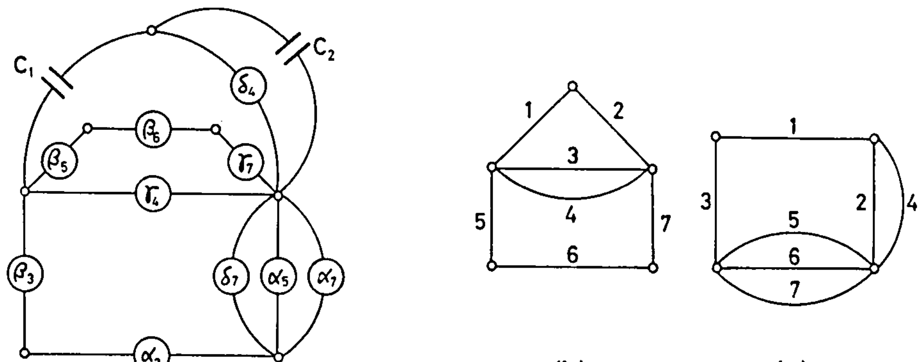


(a) (b) (c)

Fig.5-5.2 Example
(a) Original network, (b) its voltage graph and (c) its current graph.

The rank of M_0 for the network shown in Fig.5-5.3 is 13, not $2n_e=14$. There is no e-capacitor-branch for the Type 4 branch 6

in the normal maximum tree-pair $\{T_V^N : T_I^N\}$ where $T_V^N=\{C_1, C_2, 5, 6\}$ and $T_I^N=\{C_1, C_2, 5\}$.



(a) (b) (c)

Fig.5-5.3 Example
(a) Original network, (b) its voltage graph and (c) its current graph

Consider the case where the rank of $M_{\varepsilon-1}$ is $2\varepsilon n_e$ and that of M_ε is less than $2(\varepsilon+1)n_e$. Let M_ε be

$$M_{\varepsilon}=\left|\begin{array}{cccc|c|c} \theta_1 & \overset{(\varepsilon+1)}{} & & & \underline{\underline{\Delta}} & B_\varepsilon \\ \theta_2 & \theta_1 & & & & B_{\varepsilon-1} \\ & \theta_2 & & & & \\ & & \ddots & & & \\ & & & \theta_1 & & \\ & & & \theta_2 & & B_0 \end{array}\right|$$

$$M_0=B_0$$

where $B_i=\begin{vmatrix}\theta_1\\ \theta_2\end{vmatrix}$, $i=0,1,\cdots,\varepsilon-1,\varepsilon$.

The pivots in B_0 are chosen as mentioned above. Consider the pivots in B_1. The elements of C in B_1 corresponding to the e-capacitor-branches C_{e0} cannot be chosen as pivots because the elements of -1 in B_0 corresponding to those are pivots, that is, the e-capacitors are links for the tree which determines pivots in B_1. Then the graphes to be considered in choosing the pivots in B_1 are $G_V\{C_{L0}\}$ (denoted by G_{V1}) and $G_I\{C_{L0}\}$ (denoted by G_{I1}), where $C_{L0}=C_{e0}$+link capacitor-branches. In the similar manner of choosing the pivots of B_0 for a normal maximum tree pair of G_V and G_I, the pivots of B_1 can be chosen for a normal maximum tree pair of G_{V1} and G_{I1} as follows:

1) In $[B_{VC}, B_{VD}]$, the elements corresponding to the branches belonging to Type 1 and 3, and the elements corresponding to C_{L0}.

2) In C, elements to Type 2 and 4.

3) In H_{VD}, elements to Type 2 and 4.

4) In $-H_{ID}$, elements to Type 1.

5) In Q_{ID}, elements to Type 2 and 3.

6) In Q_{ID}, elements to Type 4- dependent source-branches if the fundamental loops determined by those dependent source-branches contain e-capacitor-branches.

7) In Q_{IC}, elements to Type 2 and 3 except e-capacitor-branches.

8) In 1, elements to the capacitor-branches not chosen in 7).

If there is no e-capacitor-branches, the rank of M_1 is less than $4n_e$, and in this case $\varepsilon \doteq 1$. Otherwise there exist an e-capacitor-branch for every Type 4 dependent source-branch in G_{V1} and G_{I1}. Then the rank of M_1 is $4n_e$.

Consider pivots of M_ε $\{\varepsilon=2,\cdots\}$. Since the rank of M_ε is less than $2(\varepsilon+1)n_e$, there is no e-capacitor-branch in $G_{V\varepsilon}$ and $G_{I\varepsilon}$. A normal maximum tree pair in G_{Vi} and G_{Ii} is considered when the pivots of B_i is chosen. It is a maximum tree pair in G_V and G_I. Examine the maximum tree pair. For simplicity, let us assume that there exists only one branch of Type 4, b_1, in G_V and G_I. The maximum tree pair for the pivots in B_0 is a normal maximum tree pair $\{T_V^N:T_I^N\}$. The maximum tree pair for the pivots in B_1 is a normal maximum tree pair of $G_{V1}\triangleq G_V\{C_S,C_1\}$ and $G_{I1}\triangleq G_V\{C_S,C_1\}$, where C_S is a set of link-capacitor-branches belonging to Type 1 and C_1 is an e-capacitor. The maximum tree pair for the pivots in B_2 is a normal maximum tree pair of $G_{V2}\triangleq G_V\{C_S,C_1,C_2\}$ and $G_{I2}\triangleq G_I\{C_S,C_1,C_2\}$, where C_2 is an e-capacitor in G_{V1} and G_{I1}. By continuing the above, the maximum tree pair for the pivots in $B_{\varepsilon-1}$ is a normal maximum tree pair of $G_V\{C_S,C_1,\cdots,C_{\varepsilon-1}\}$ and $G_I\{C_S,C_1,\cdots,C_{\varepsilon-1}\}$, where C_i is an e-capacitor in $G_V\{C_S,C_1,\cdots,C_{i-1}\}$ and $G_I\{C_S,C_1,\cdots,C_{i-1}\}$. Since the rank of M_ε is less than $2(\varepsilon+1)n_e$, there exists no capacitor-branch C_ε such that a normal maximum tree pair of $G_V\{C_S,C_1,\cdots,C_{\varepsilon-1},C_\varepsilon\}$ and $G_I\{C_S,C_1,\cdots,C_{\varepsilon-1},C_\varepsilon\}$ is a maximum tree pair of G_V and G_I. Consequently in determining the pivots of M_ε, a normal maximum tree pair of G_V and G_I is chosen to determine the pivots of B_0, and a primitive maximum tree pair in Definition 5-2.3, to determine those of $B_{\varepsilon-1}$.

Meanwhile two graphes are uniquely partitioned into the pair subgraphes for an arbitrary maximum tree pair[21]. The partitioned subgraphes have one to one correspondence to the branches belonging to Type 3 or 4. Then we obtain

Lemma 5-5.2

A minimal index of the column dependence ε_k of the pencil (5-5.2) is given by

$$\varepsilon_k = |T_V^N(C)| - |T_I^P(C)|$$

for the partitioned graph of G_V and G_I which has only one branch belonging to Type 4.

Consider the row dependence of the pencil (5-5.2). For this, the column dependence of the pencil $\theta_1' + p\theta_2'$. The matrix M_0 for $\theta_1' + p\theta_2'$ is written as

$$M_0 = \begin{vmatrix} 0, & 0 & ,B'_{VC}, & 0 \\ 0, & H'_{VD}, & B'_{VD}, & 0 \\ -1, & 0 & , 0 & ,Q'_{IC} \\ 0, & H'_{ID}, & 0 & ,Q'_{ID} \\ C, & 0 & , 0 & , 0 \\ 0, & 0 & , 0 & , 0 \end{vmatrix}.$$

Let us choose pivots of a minor determinant of M_0 as follows:

1) In $\begin{vmatrix} Q'_{IC} \\ Q'_{ID} \end{vmatrix}$, elements corresponding to the branches belonging to Type 2 and 3.

2) In H'_{ID}, elements to Type 1 and 4.

3) In H'_{VD}, elements to Type 2 and 3.

4) In 1, elements to Type 1 and 4.

5) In C, elements to Type 2 and 3.

6) In $\begin{vmatrix} B'_{VC} \\ B'_{IC} \end{vmatrix}$, elements to Type 1 and the capacitor-branches which is in the *loop(b)* in G_V (where b is a branch belonging to Type 3).

Between the above pivot-terms and those for the case of $\theta_1+p\theta_2$, there are some correspondences shown in Table 5-5.2.

Then we obtain

$\theta'_1+p\theta'_2$	$\theta_1+p\theta_2$
Type 3	Type 4
G_V	G_I
G_I	G_V

Table 5-5.2
Correspondence of pivots of M_0.

Lemma 5-5.3

A minimal index η_k of the row dependence of the pencil (5-5.2) is given by

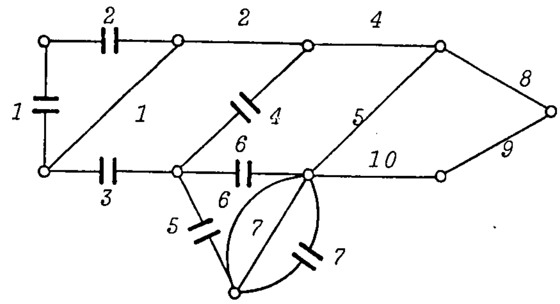
$$\eta_k = |T_I^N(C)| - |T_V^P(C)|$$

for the partitioned graph of G_V and G_I which has only one branch belonging to Type 3.

Examine a canonical form of the network-equation for the network shown in Fig.5-5.4, where the voltage and the current graphs are shown in Fig.5-5.4 (a) and (b), respectively. The graphes G_{V1}, G_{I1}, G_{V2} and G_{I2} are derived from G_V and G_I for the branches belonging to Type 4. The graphs G_{V3} and G_{I3} are partitioned for the branches belonging to Type 3. The graphs G_{V4} and G_{I4} has common trees.

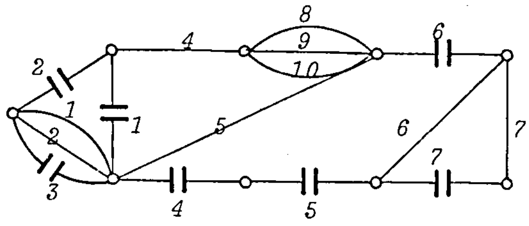
The network-equation for the network shown in Fig.5-5.4 is strictly equivalent to

$$\left|\begin{array}{c|c} \begin{matrix} 0,p,1,0,0,0 \\ 0,0,p,1,0,0 \\ 0,0,0,0,p,0 \\ 0,0,0,0,1,p \\ 0,0,0,0,0,1 \end{matrix} & 0 \\ \hline 0 & A_0+pB_0 \end{array}\right| ,$$



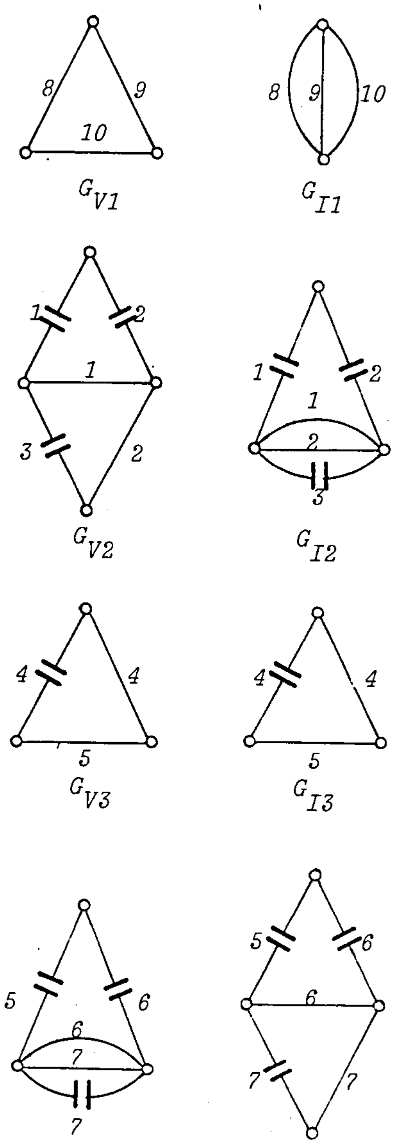
G_V

(a)



G_I

(b)



(c) (d)

Fig.5-5.4 Example

where A_0+pB_0 is a regular pencil.

The method of obtaining a canonical form of a network-equation is given in this section. To obtain a canonical form of the regular pencil A_0+pB_0 is to obtain a state equation. It is discussed in Section 5-4.

5-6 CONCLUDING REMARKS

In Section 5-2, the induced digraph induced from the voltage and current graphs is defined, and the properties of the digraph are considered. They are useful in formulating the state equation.

In Section 5-3, the solvability and the order of complexity are considered.

In Section 5-4, a state equation

whose state variables are the voltage across the twig-capacitor for a normal common tree is formulated. It may include higher order-derivatives of the inputs than the first order.
The relation between the higher order-derivatives and the network-topology is studied.

In Section 5-5, a canonical form of a network-equation with no unique solution is studied by means of 'matrix-pencil'. It gives the properties of the network. It may be useful for the piecewise linear analysis of a nonlinear network such as the network-topology of some piecewise linearized networks is different from that of others.

The considerations in this chapter deal with the network-topology only. However the problems on the solvability and the formulation of a state equation depend not only on the network-topology but also the network-element-values. The problems considering the both are not studied. They may have no fruitful result.

CHAPTER 6 LINEAR NETWORKS CONTAINING PERIODICALLY OPERATED SWITCHES (9)-(15)

6-1 INTRODUCTION

Periodically interrupted electric networks (networks containing periodically operated switches) play an important role in many circuit applications[28][33].

Methods for analizing such networks have been provided by many authors[28]-[32]. S.Hayashi[28] analized such circuits by means of matric-operational calculus and introduced two kinds of initial values. W.R.Bennet[29] gave steady state solutions with sinusoidal inputs. Y.Sun and I.T.Frish[30] analized circuits containing switches, resistors and capacitors, and succeeded in obtaining circuits with larger time constants than those containing no switches. A.Fetweis[31] gave an analysis by means of z-transformation, and M.L.Liou[32], by means of state variables.

The above mentioned methods are applicable to the networks with certain restrictions on the inputs [28] or on the network-structures[29][30]. But the formal relations between two kinds of initial values and the stability problem (that is, the problem whether the circuits have steady state solutions or not) are not studied. The change of network-topology due to switch-operations is not discussed.

In Section 6-2, changes of network-topology due to switch-operations are studied.

In Section 6-3, analysis of networks containing periodically operated switches is studied by state variable-approach.

In Section 6-4, the initial values of the second kind[28] are derived formally from those of the first kind, in case of RLC, RLCT and RCG networks. In active networks, the relation between the initial values of the first and second kinds is discussed.

In Section 6-5, the mathematical meaning of the two kinds of initial values is discussed.

In Section 6-6, the problem on the stability is studied.

6-2 NETWORK-TOPOLOGY AND RESTRICTIONS

6-2.1

The Fig.6-2.1 shows a general network containing periodically operated switches, where the switches represented by S are closed and opened with a period T. By switch-operations, n circuit-modes are produced periodically. Each circuit-mode is produced again after the period T. and its interval is called a stage. (See Fig.6-2.2)

The graph corresponding to the network containing periodically operated switches is denoted by G^*. Note that the graph G^* contains branches corresponding to switches (switch-branches).

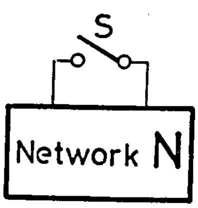


Fig.6-2.1
Network containing periodically operated switches.

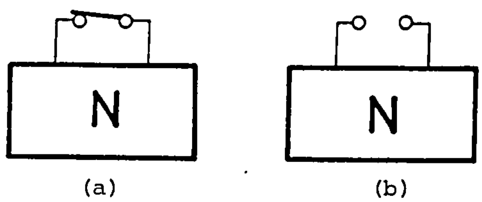


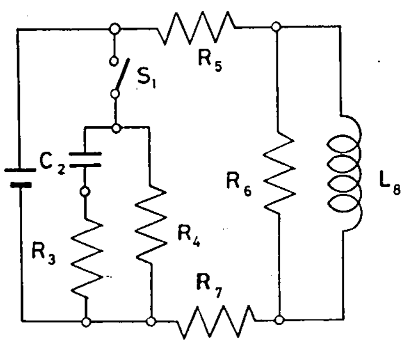
Fig.6-2.2
(a) Circuit-mode; switch is closed.
(b) Circuit-mode; switch is opened.

The graph obtained from G^* by contracting voltage source-branches and deleting current source-branches is denoted by G.

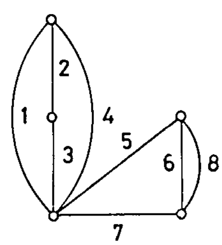
In general, the graph G may be separable, *i.e.* there may be some network-elements whose currents and voltages do not depend on any switch-operation. In our analysis, only a connected component of G containing some switch-branches may be considered. For example, the graph G corresponding to the network shown in Fig.6-2.3(a) is shown in Fig.6-2.3(b). Then only a network shown in Fig.6-2.3(c) is considered.

The symbol $G(S_1,\cdots,S_k)$ denotes the graph corresponding to the network, where $S_i=1$"$S_i=0$" if the switch S_i is closed"opened", and k is the number of switches. The symbol G_i denotes the graph corresponding to the i-th circuit mode.

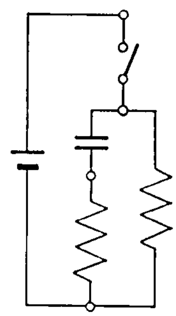
In the state variable-approach, a normal tree, a maximum proper tree or a normal common tree is chosen for every circuit mode if the circuit mode is of an RLC; RLCT"RCG" or active network, respectively.



(a)



(b)



(c)

Fig.6-2.3 Example of separable graph

(a) Original network.

(b) Its corrresponding graph.

(c) Network to be considered.

Assume that every circuit mode is of an RLC network. Let us choose a normal tree (denoted by T_s) in $G(1,\cdots,1)$. A normal tree T_o in $G(0,\cdots,0)$ can be obtained such that it includes T_s. Then a normal tree (denoted by T_i) in G_i containing all twigs in T_s and no links for T_o can be chosen. Such a tree is called a basic tree for T_s and T_o. Then the voltages across the twig-capacitors in T_s and the currents through the link-inductors for T_o can always be state variables in every circuit mode. However, the state variables in some circuit mode are not only these voltages and currents since generally the order of complexity of one circuit mode may be different from that of another.

Definition 6-2.1

Let T be a basic tree of $G(1,\cdot\cdot,1,\overset{i}{0},1,\cdot\cdot,1)$ for T_s and T_o. If the number of the twig-capacitors in T is not equal to that in T_s, the switch S_i is called a C-switch, if the number of the link-inductors in $\bar{T}$ is not equal to that in $\bar{T}_o$, the switch S_i is called an L-switch, and otherwise, the switch S_i is called an R-switch.

From Definition 6-2.1, if all switches are R-switches, the order of complexity and a set of state variables do not vary in switch-operations.

Consider how the network-topology changes by switch-operation if the network contains only one R-switch S_1'. Then the characteristic part of the fundamental loop matrix of G is given by

$$\begin{array}{c} \\ S \\ R \\ y \\ L \end{array}\begin{array}{c|ccc} (S_1) & (C) & (G) & (\Gamma) \\ \hline 0 & \mathsf{F}_{SC}, & 0, & 0 \\ \mathsf{f}_{Rx} & \mathsf{F}_{RC}, & \mathsf{F}_{RG}, & 0 \\ f_{yx} & \mathsf{f}_{yC}, & \mathsf{f}_{yG}, & 0 \\ \mathsf{f}_{Lx} & \mathsf{F}_{LC}, & \mathsf{F}_{LG}, & \mathsf{F}_{L\Gamma} \end{array}, \qquad (6\text{-}2.1)$$

where the switch-branch S_1 is a twig. The subscripts x and y denote switch S_1 and the resistor which becomes a twig when S_1 becomes a link. The matrix obtained from (6-2.1) by deleting the column for S_1 is the characteristic part of the loop matrix of $G(1)$.

Then the characteristic part of the fundamental loop matrix, when S_1 is chosen as a link, is

$$\begin{array}{c} \\ (S) \\ (R) \\ (L) \\ (S_1) \end{array}\begin{array}{cccc} (C) & (G) & (y) & (\Gamma) \\ \left|\begin{array}{c} \mathsf{F}_{SC} \\ \mathsf{F}_{RC}-f_{yx}\mathsf{f}_{Rx}\mathsf{f}_{yC} \\ \mathsf{F}_{LC}-f_{yx}\mathsf{f}_{Lx}\mathsf{f}_{yC} \\ -f_{yx}\mathsf{f}_{yC} \end{array}\right. & \begin{array}{c} 0 \\ \mathsf{F}_{RG}-f_{yx}\mathsf{f}_{Rx}\mathsf{f}_{yG} \\ \mathsf{F}_{LG}-f_{yx}\mathsf{f}_{Lx}\mathsf{f}_{yG} \\ -f_{yx}\mathsf{f}_{yG} \end{array} & \begin{array}{c} 0 \\ f_{yx}\mathsf{f}_{yG} \\ f_{yx}\mathsf{f}_{yG} \\ f_{yx} \end{array} & \left.\begin{array}{c} 0 \\ 0 \\ \mathsf{F}_{L\Gamma} \\ 0 \end{array}\right| \end{array}, \qquad (6\text{-}2.2)$$

by Lemma A-3.2. (See Appendix A-3.2.) The matrix obtained from (6-2.2) by deleting the row for S_1 is the characteristic part of the loop matrix of $G(0)$.

In the network containing more than one R-switch, the variations in network-topology caused by switch-operations are complicated unless the network satisfies the following assumption.

Assumption 6-2.1

There is no intersection of S_{l1} and S_{l2}, where S_{l1}"S_{l2}" is a subset of links $\bar{T}_s$ each of which can become a twig of a tree for

$G(1,\cdots,1,\overset{i}{0},1,\cdots 1)$"$G(1,\cdots,1,\overset{j}{0},1,\cdots,1)$"$\{i,j=1, \quad . \; i\neq j\}$.

Consider a network containing n R-switches. The characteristic part of the fundamental loop matrix for T_s is written as

$$\begin{vmatrix} F_{SC}, & 0, & 0 \\ F_{RC}, & F_{RG}, & 0 \\ F^{1}_{RC}, & F^{1}_{RG}, & 0 \\ f^{1}_{RC}, & f^{1}_{RG}, & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ F^{n}_{RC}, & F^{n}_{RG}, & 0 \\ f^{n}_{RC}, & f^{n}_{RG}, & 0 \\ F_{LC}, & F_{LG}, & F_{L\Gamma} \\ F^{1}_{LC}, & F^{1}_{LG}, & F^{1}_{L\Gamma} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ F^{n}_{LC}, & F^{n}_{LG}, & F^{n}_{L\Gamma} \end{vmatrix} . \qquad (6\text{-}2.3)$$

From the matrix (6-2.3), the characteristis part of the fundamental loop matrix when S_i and S_j are links is written as the following matrix (6-2.4), where the elements for switches are eliminated, and the supperscript i of F^{i}_{RC} denotes a characteristic part corresponding to the resistor-branches for which the fundamental loops contain the switch S_i.

$$
\left|\begin{array}{c|c|c|c|c}
F_{SC} & 0 & 0 & 0 & 0 \\
\hline
F_{RC} & F_{RG} & & & \\
F_{RC}^{1} & F_{RG}^{1} & & & \\
\vdots & \vdots & & & \\
f_{RC}^{i-1} & f_{RG}^{i-1} & & & \\
F_{RC}^{i}-\left|\begin{array}{c}1\\ \vdots\\ 1\end{array}\right| f_{RC}^{i} & F_{RG}^{i}-\left|\begin{array}{c}1\\ \vdots\\ 1\end{array}\right| f_{RG}^{i} & \left|\begin{array}{c}-1\\ \vdots\\ -1\end{array}\right| & & 0 \\
F_{RC}^{i+1} & F_{RG}^{i+1} & & & \\
\vdots & \vdots & & & \\
f_{RC}^{j-1} & f_{RG}^{j-1} & & & \\
F_{RC}^{j}-\left|\begin{array}{c}1\\ \vdots\\ 1\end{array}\right| f_{RC}^{j} & F_{RG}^{j}-\left|\begin{array}{c}1\\ \vdots\\ 1\end{array}\right| f_{RG}^{j} & & \left|\begin{array}{c}-1\\ \vdots\\ -1\end{array}\right| & \\
F_{RC}^{j+1} & F_{RG}^{j+1} & & & \\
\vdots & \vdots & & & \\
f_{RC}^{n} & f_{RG}^{n} & & & \\
\hline
F_{LC} & F_{LG} & & & F_{L\Gamma} \\
F_{LC}^{1} & F_{LG}^{1} & & & F_{L\Gamma}^{1} \\
\vdots & \vdots & & & \vdots \\
F_{LC}^{i}-\left|\begin{array}{c}1\\ \vdots\\ 1\end{array}\right| f_{RC}^{i} & F_{LG}^{i}-\left|\begin{array}{c}1\\ \vdots\\ 1\end{array}\right| f_{RG}^{i} & \left|\begin{array}{c}-1\\ \vdots\\ -1\end{array}\right| & & F_{L\Gamma}^{i} \\
\vdots & \vdots & & & \vdots \\
F_{LC}^{j}-\left|\begin{array}{c}1\\ \vdots\\ 1\end{array}\right| f_{RC}^{j} & F_{LG}^{j}-\left|\begin{array}{c}1\\ \vdots\\ 1\end{array}\right| f_{RG}^{j} & & \left|\begin{array}{c}-1\\ \vdots\\ -1\end{array}\right| & F_{L\Gamma}^{j} \\
\vdots & \vdots & & & \vdots \\
F_{LC}^{n} & F_{LG}^{n} & & & F_{L\Gamma}^{n}
\end{array}\right| . \tag{6-2.4}
$$

6-2.2

Consider a network satisfying Assumption 6-2.1. Then its graph satisfies;

Condition 6-2.1

There exists a tree such that several specified branches are twigs and there is no fundamental loop containing two or more of the specified branches.

The dual condition of Condition 6-2.1 is

Condition 6-2.2

There exists a tree such that several specified branches are links and there is no fundamental cut-set containing two or more of the specified branches.

Assume that the number of specified branches is two. We have some definitions.

Definition 6-2.2

The branches specified to be twigs are called r-branches (reserved branches). The nodes which touch r-branches are called r-nodes.

Definition 6-2.3

A tree (containing r-branches) is called a p-tree (partitioning tree) if every fundamental loop with respect to the tree contains less than two of r-branches.

Definition 6-2.4

A path from a node i to j which does not contain r-branches and r-nodes except the r-nodes i and j is denoted by $p(i,j)$.

A node is called a p-node if it is not an r-node and if there exist paths from it to more than three r-nodes containing no r-branch.

A path $p(i,j)$ containing no p-node is denoted by $d\text{-}p(i,j)$ (direct path).

From the difinitions, we obtain

Lemma 6-2.1

If a node touching a branch is a p-node, the other node touching the branch is an r-node or a p-node.

Proof: If the node is not an r-node, there exists a path from the node to the p-node. Then the node is a p-node.

Q.E.D.

Consider the case where an r-node touching an r-branch 1 is identical to an r-node touching r-branch 2. Without loss of generality, the other r-nodes can be assumed to be labeled with $1'$ and $2'$. Then we obtain

Theorem 6-2.1

In the case where the r-node 1 is identical to the r-node 2, there exists a p-tree if and only if there exist no $d\text{-}p(1',2')$.

Proof: Either there is a $p(1',2')$ or not.

i) If there is no $p(1',2')$, then the graph is separable and the r-node 1"2" is a cut-node. Then any tree containing the r-branches 1 and 2 is a p-tree.

ii) If there exists a $p(1',2')$, there exists at least one p-node in the $p(1',2')$. Then a tree containing the r-branches 1 and 2, such that every p-node is connected to the r-node 1 with a path in it even if the r-branches are deleted, can be chosen. If a node touching a link for the tree is a p-node, the other node touching the link is a p-node or an r-node from Lemma 6-2.1. Then the two nodes are connected in the tree if either of the two r-branches is deleted. Then the fundamental loop for the link does not contain both of the r-branches. If either node touching a link in $p(1',2')$ were not a p-node, (when the fundamental loop for the link contains both of the r-branches,) there would exist a $d\text{-}p(1',2')$. This contradicts the assumption, and therefore the tree is a p-tree.

If there exists a $d\text{-}p(1',2')$, the fundamental loop for a link in the $d\text{-}p(1',2')$ with respect to any tree contains both of the r-branches. Then there does not exist a p-tree.

Q.E.D.

Consider the case where all r-nodes are different. The following theorem covers the case since the r-nodes can be arbitrarily labeled.

Theorem 6-2.2

Suppose that all r-nodes are different. i) If there exist all $d\text{-}p(1,2)$, $d\text{-}p(1',2')$, $d\text{-}p(1',2)$ and $d\text{-}p(1,2')$, or ii) if there exists $d\text{-}p(1,2)$ and $d\text{-}p(1',2')$ and neither of $p(1',2)$ and $p(1,2')$, then there is no p-tree. The converse is also true.

Proof: In the case where there exist all *d-p(1,2)*, *d-p(1',2')*, *d-p(1',2)* and *d-p(1,2')*, even if the r-branches are deleted in any tree *T* containing the r-branches, there exists a path from the r-node *1* to *2*, from *1'* to *2'*, from *1'* to *2* or from *1* to *2'* in the subtree *T*-{r-branches}. Assume that there exists a path from *1* to *2*. A fundamental loop for a link in the *d-p(1',2')* contains both of the r-branches. Therefore there exists no p-tree. Similarly there exists a path from *1'* to *2'*, from *1'* to *2* or from *1* to *2'* in *T*-{r-branches}.

In the case where there exist *d-p(1,2)* and *d-p(1',2')* and there exists neither *p(1',2)* nor *p(1,2')*, there exists either a path from the r-node *1* to *2* or from *1'* to *2'* in a subtree where the r-branches are deleted in any tree containing the r-branches. Assume that there exists a path from *1* to *2*. Then a fundamental loop for a link in the *d-p(1',2')* contains both of the r-branches. Therefore there exists no p-tree.

The cases which are not covered by the case where there exist all *d-p(1,2)*, *d-p(1',2')*, *d-p(1',2)* and *d-p(1,2')*, and the case where there exist *d-p(1,2)* and *d-p(1',2')* and there does not exist either *p(1',2)* or *p(1,2')* are as follows;

Case 1) None of *d-p(1,2)*, *d-p(1',2')*, *d-p(1',2)* and *d-p(1,2')* exists.

Case 2) There exists only *d-p(1,2)* and there is no other d-path.

Case 3) There exist *d-p(1,2)* and *d-p(1,2')*, and there does not exist another d-path.

Case 4) There exist *d-p(1,2)*, *d-p(1,2')* and *d-p(1',2)*, and there is no *d-p(1',2')*.

Case 5) There exist $d\text{-}p(1,2)$ and $d\text{-}p(1',2')$, and there exists either $p(1',2)$ or $p(1,2')$.

We prove that there exists a p-tree in the five cases.

Case 1) There exists either of the paths $p(i,j)$ $\{i=1,1'\ j=2,2'\}$ since the graph considered is connected. Then there exists at least a p-node, and a node in $p(i,j)$ is either an r-node or a p-node. There exists a tree containing both of the r-branches, such that there is a path from a p-node to another in the tree even if either of the r-branches is deleted. Then a fundamental loop for a link touching a p-node does not contain both of the r-branches. If neither of nodes touching a link were a p-node and if the fundamental loop for the link contained both of the r-branches, there would exist a $d\text{-}p(i,j)$ $\{i=1$ or $1'$ $j=2$ or $2'\}$. This contradicts the assumption of Case 1). There exists a p-tree.

Case 2) Since there exists a path from a p-node to r-node 1 or 2 which does not contain r-nodes except the terminal nodes, there exists a tree containing both of the r-branches and all branches of a $d\text{-}p(1,2)$, such that there exists a path from a p-node to another p-node in the tree even if either of the r-branches is deleted. Then a fundamental loop for a link touching a p-node does not contain both of the r-branches. Assume that neither node touching a link is a p-node. The link is either contained in a $p(1,2)$ or not. In the former, the fundamental loop for the link could not contain either of the r-branches since all branches in a $d\text{-}p(1,2)$ are twigs. In the latter, the fundamental loop for the link would contain both of the r-branches. Then there would exist a $d\text{-}p(1',2')$, $d\text{-}p(1,2')$ or $d\text{-}p(1',2)$. This contradicts the assumption of Case 2). Thus there exists a p-tree.

Case 3) There exists a tree containing both of the r-branches and all branches in a *d-p(1,2')*, and a path from a p-node to another p-node exists in the tree even if either of the r-branches is deleted. Then a fundamental loop for a link touching a p-node does not contain both of the r-branches. If the link is contained in a *d-p(1,2)* or another *d-p(1,2')*, the fundamental loop for the link does not contain both of the r-branches since all branches of a *d-p(1,2')* are twigs. If the link is not contained in a *d-p(1,2)* or a *d-p(1,2')* and if the fundamental loop for the link contains both of the r-branches, then there exists a *d-p(1,2)*. This contradicts the assumption of Case 3). There exists a p-node.

Case 4) There exists a tree containing both of the r-branches and all branches in a *d-p(1,2)*, and a path from a p-node to another p-node exists in the tree even if either of the r-branches is deleted. Similar auguments to those in Case 1)-3) lead to that the tree is a p-tree.

Case 5) There exists a tree containing both of the r-branches and all branches of a *p(1',2)* or a *p(1,2')*, and a path from a p-node to another p-node exists in the tree even if either of the r-branches is deleted. Similar auguments to those in Case1)-4) lead to that the tree is a p- tree.

Q.E.D.

The problem whether there exists a p-tree or not is solved for a graph where the number of r-branches is two. The problem where the number of r-branches is more than two is considered. This problem is not solved. But a heuristic algorithm for a p-tree is obtained in the next paragraph. The proof of the algorithm is not given.

In order to consider condition 6-2.2, we have some definitions.

Definition 6-2.4

The branches specified to be links are called s-branches and the nodes which touch s-branches are called s-nodes.

Definition 6-2.5

A tree (containing no s-branches) is called a d-tree if every fundamental cut-set with respect to it contain less than two s-branches.

For a graph with two s-branches S_1 and S_2, we have

Theorem 6-2.3

A tree which contains no s-branch is denoted by T. Assume that there exists a twig t_0 such that the fundamental cut-set for t_0 with respect to T (*cut-set*(t_0)) contains two s-branches. (If the tree T does not satisfy the assumption, it is a d-tree.) If the link l_1 and the twig t_1 exists such that

i) The link l_1 is in *cut-set*(t_0),

ii) The twig t_1 is in *loop*(l_1), and *cut-set*(t_1) contains only one s-branch (*e.g.* S_1),

iii) There is no twig in *loop*(l_1) such that the fundamental cut-set for the twig contains S_2 but does not contain S_1, then the number of the twigs in the tree $T-t_1+l_1$ such that the fundamental cut set for each of the twigs with respect to $T-t_1+l_1$ contains two s-branches is less than that in T.

Proof: Let the fundamental cut-set for a twig t with respect to T"$T-t_1+l_1$" be $C(t)$"$C_1(t)$". Then if a twig t is in $loop(l_1)$ with respect to T but is not t_1, (a) $C_1(t)=C(t)+C(t_1)$. Otherwise (b) $C_1(t)=C(t)$, and (c) $C_1(t_1)=C(t_1)$. From iii), $C_1(t)$ in (a) does not contain two r-branches. Then the theorem is proved.

Q.E.D.

Corollary

If the branches as the link l_1 and the twig t_1 do not exists, and if T is not a d-tree, then there is no d-tree.

6-2.3 ALOGORITHM FOR p-TREES AND d-TREES

i) The number of r-branches is two.

By Theorem 6-2.1 and 6-2.2, it can be determined whether there exists a p-tree or not. If there exists a p-tree, it can be obtained by choosing a tree as described in the proofs of the theorems.

ii) The number of s-branches is two.

By Theorem 6-2.3, it can be determined whether there exists a d-tree or not. If there exists a d-tree, a d-tree can be obtained by tree-transformations mentioned in Theorem 6-2.3

iii) The number of r-branches is more than two.

In this case, we have no necessary and sufficient condition for the existence of a p-tree. However, the following theorem gives a necessary condition.

Theorem 6-2.4

Let n_n and n_b denote the numbers of nodes and branches in a graph, respectively. Let M_{ci} be minimum number of branches in a cut-set among the cut-sets containing only one r-branch r_i. Then if there exists a p-tree, the following inequality holds;

$$n_b - \sum_{r_i \in R} M_{ci} \geq n_n - (|R|+1), \qquad (6\text{-}2.5)$$

where R *is* the set of r-branches.

Proof: Since there exists a p-tree (*i.e.* there is no intersection among the cut-sets for r-branches), let the number of the branches of the fundamental cut-set of the r-branches r_i for the p-tree be F_{ci}, then

$$n_b - \sum_{r_i \in R} M_{ci} \geq n_b - \sum_{r_i \in R} F_{ci}. \qquad (6\text{-}2.6)$$

Then the right hand of Eq.(6-2.6) is not less than the rank of G[r-branches]. Consequently

$$n_b - \sum_{r_i \in R} F_{ci} \geq n_n - (|R|+1).$$

The theorem is proved.

Q.E.D.

We have a hueristic algorithm of obtaining a p-tree. It is shown in Appendix VI.

6-2.4 CONCLUDING REMARK

The similar problems concerning p-tree or d-tree are

i) the problem of choosing a tree such that there are dummy branches in the different subgrahs for the fundamental loops[37].

ii) the dual problem of i).

iii) the problem of maximizing the sparsity of the admittance matrix Y[38].

6-3 ANALYSIS OF NETWORKS CONTAINING PERIODICALLY OPERATED SWITCHES

Method of analizing such networks have already been provided. Matric operational calculus[28], z-transformation[31] and state variable-approaches[32] are examples of such methods. The state variable-approaches fit the computer-aided analysis and have some informations of the network topology. Here a state variable-approach to the analysis of networks containing periodically operated switches is studied. Such networks are assumed to satisfy the following Restriction A.

Restriction A

Every circuit mode satisfies Assumption A in Chapter 2, and does not contain impulsive voltage- or current-sources.

For an RLC network, as mentioned in Section 6-2, the graph corresponding to every circuit mode has a normal tree , which contains all twigs of a normal tree T_s in $G(1,\cdots,1)$ and no links of a normal tree T_o in $G(0,\cdots,0)$. Therefore a set of state variables for some circuit mode can be chosen so that the set covers a set of state variables for the circuit mode whose order of complexity is minimum among the all circuit modes. This state variables can be covered by state variables of the circuit mode whose order of complexity is maximum among all the circuit modes.

Let us consider a network with two circuit modes. The period of the modes is denoted by T. Assume that a set of state variables for the first circuit mode covers that for the second.

The state equations for the first and the second circuit modes for the n-th stage may be obtained, respectively, as

$$\dot{x}_{1,n}(t)=A_1 x_{1,n}(t)+B_1 u(t+nT), \quad \text{at } nT<\tau<nT+t_1, \tag{6-3.1}$$

where $t=\tau-nT$, and

$$\dot{x}_{2,n}(t)=A_2 x_{2,n}(t)+B_2 u(t+nT+t_1), \quad \text{at } nT+t_1<\tau<(n+1)T, \tag{6-3.2}$$

where $t=\tau-(nT+t_1)$. The vectors $x_{i,n}(t)$ $\{i=1,2\}$ and $u(t)$ are the state and the input vector, respectively. The matrices A_i and B_i are constant.

Assume that B_i contains no derivative term. (It wil be shown in the following Section 6-4 that the assumption is reasonable.) Then the solution of Eq.(6-3.1) and (6-3.2) are obtained, respectively as

$$x_{1,n}(t)=e^{tA_1}x_{1,n}(+0)+e^{tA_1}\int_{nT}^{nT+t} e^{-(\tau-nT)A_1}B_1 u(\tau)d\tau, \tag{6-3.3}$$

$$x_{2,n}(t)=e^{tA_2}x_{2,n}(+0)+e^{tA_2}\int_{nT+t_1}^{(n+1)T} e^{-(\tau-nT-t_1)A_2}B_2 u(\tau)d \ , \tag{6-3.4}$$

where $x_{i,n}(+0)$ $\{i=1,2\}$ are the initial value vectors of the second kind.

Consider the input functions,

$$u(\tau)=ue^{j\overline{\omega\tau+\phi}}, \tag{6-3.5}$$

where u is a constant vector. Then Eq.(6-3.3) and Eq.(6-3.4) become, respectively,

$$x_{1,n}(t)=e^{tA_1}x_{1,n}(+0)+(j\omega 1-A_1)^{-1}(e^{j\omega t}1-e^{tA_1})B_1 ue^{j\overline{\omega nT+\phi}}, \tag{6-3.7}$$

$$x_{2,n}(t)=e^{tA_2}x_{2,n}(+0)+(j\omega 1-A_2)^{-1}(e^{j\omega t}1-e^{tA_2})B_2 ue^{j(\overline{\omega nT+t_1}+\phi)}, \tag{6-3.8}$$

i) Since the set of state variables for the first circuit mode covers that of the second, some voltages across the capacitors and/or some currents through the inductors may be state variables but not those for the second circuit mode. However, such voltages or currents for the second circuit mode can be expressed in terms of the state variables. A network-element corresponding to the j-th element of $x_{1,n}(t)$ can be identical to that of $x_{2,n}(t)$.

ii) The initial value vectors of the second kind, $x_{1,n}(+0)$ and $x_{2,n}(+0)$, can be derived from $x_{2,n}(t_2)$ and $x_{1,n-1}(t_1)$, respectively. (This shall be mentioned in Section 6-4.)

From the discussion i) and ii), Eq.(6-3.7) and (6-3.8) can be written, respectively, as

$$x_{1,n}(t)=H_1(t)x_{2,n-1}(t_2)+\boldsymbol{l}_1(t)e^{j\omega nT}, \qquad (6\text{-}3.9)$$

$$x_{2,n}(t)=H_2(t)x_{1,n}(t_1)+\boldsymbol{l}_2(t)e^{j\omega\overline{nT+t_1}}, \qquad (6\text{-}3.10)$$

where $T=t_1+t_2$.

The above equations lead to

$$x_{1,n}(t_1)=D_1x_{1,n-1}(t_1)+S_1e^{j\omega nT}$$

$$x_{2,n}(t_2)=D_2x_{2,n-1}(t_2)+S_2e^{j\omega\overline{nT+t_1}},$$

where $D_1=H_1(t_1)H_2(t_2)$, $\quad S_1=H_1(t_1)\boldsymbol{l}_2(t_2)e^{j\omega t_2}+\boldsymbol{l}_1(t_1)$

$D_2=H_2(t_2)H_1(t_1)$, $\quad S_2=H_2(t_2)\boldsymbol{l}_1(t_1)e^{-j\omega t_1}+\boldsymbol{l}_2(t_2)$.

Consequently, $x_{1,n}(t_1)$ and $x_{2,n}(t_2)$ are expressed by $x_{1,1}(t_1)$ and $x_{2,1}(t_2)$, respectively, as

$$x_{1,n}(t_1)=D_1^{n-1}x_{1,1}(t_1)+(1-D_1e^{-j\omega T})^{-1}(1e^{j\omega nT}-D_1^{n-1})S_1, \qquad (6\text{-}3.11)$$

$$x_{2,n}(t_2)\ D_2^{n-1}x_{2,1}(t_2)+(1-D_2e^{-j\omega T})^{-1}(1e^{j\omega nT}-D_2^{n-1})S_2e^{j\omega t_1}. \qquad (6\text{-}3.12)$$

The vectors $x_{i,n}(t_1)$ $\{i=1,2\}$ can be partitioned into three vectors as

$$x_{i,n}(t_i)=\{x_{i,n}(t_i)\}_t+\{x_{i,n}(t_i)\}_o+\{x_{i,n}(t_i)\}_s, \tag{6-3.13}$$

where

$$\{x_{i,n}(t_i)\}_t=D_i^{n-1}x_{i,1}(t_i)$$

$$\{x_{1,n}(t_1)\}_o=-(1-D_1e^{-j\omega T})D_1^{n-1}S_1$$

$$\{x_{2,n}(t_2)\}_o=-(1-D_2e^{-j\omega T})D_2^{n-1}S_2e^{j\omega t_1}$$

$$\{x_{1,n}(t_1)\}_s=(1-D_1e^{-j\omega T})^{-1}S_1e^{j\omega nT}$$

$$\{x_{2,n}(t_2)\}_s=(1-D_2e^{-j\omega T})^{-1}S_2e^{j\omega\overline{nT+t_1}}. \tag{6-3.14}$$

The terms $\{x_{i,n}(t_i)\}_t$, $\{x_{i,n}(t_i)\}_o$ and $\{x_{i,n}(t_i)\}_s$ are called the transient state terms for the initial values, the transient state terms for the input voltages and currents, and the steady state terms, respectively.

If $D_1^n\to 0$ and $D_2^n\to 0$ as $n\to\infty$, then $x_{i,n}(t_i)$ contain only steady state terms at $n=\infty$.

Let $\{x_{1,n}(t)\}_s$ and $\{x_{2,n}(t)\}_s$ denote steady state solutions of the network, then $\{x_{1,n}(t)\}_s$ and $\{x_{2,n}(t)\}_s$ are given, respectively, as

$$\{x_{1,n}(t)\}_s=\{H_1(t)(1-D_2e^{-j\omega T})^{-1}S_2e^{j\omega t_1}+\boldsymbol{l}_1(t)\}e^{j\omega nT}$$
$$\triangleq f_1(t)e^{j\omega nT} \tag{6-3.15}$$

$$\{x_{2,n}(t)\}_s=\{H_2(t)(1-D_1e^{-j\omega T})^{-1}S_1e^{-j\omega t_1}+\boldsymbol{l}_2(t)\}e^{j\omega\overline{nT+t_1}}$$
$$\triangleq f_2(t)e^{j\omega nT}. \tag{6-3.16}$$

The time origin in the function $x_{i,n}(t)$ is at the start of the i-th circuit mode for the n-th stage. Here, the time origin is moved at $\tau=0$. Let $x(\tau)$ denote the steady state solution of the network as

$$x(\tau)=\begin{cases}\{x_1(\tau-nT)\}_s & \text{if } nT\leqq\tau<n\ +t_1\\ \{x_2(\tau-nT-t_1)\}_s & \text{if } nT+t_1\leqq\tau<(n+1)T.\end{cases} \qquad (6\text{-}3.17)$$

From Eq. (6-3.16), we obtain

$$x(\tau)=\begin{cases}f_1(\tau-nT)e^{j\omega nT} & nT\leqq\tau<nT+t_1\\ f_2(\tau-nT-t_1)e^{j\omega nT} & nT+t_1\leqq\tau<(n+1)T.\end{cases} \qquad (6\text{-}3.18)$$

From (6-3.15) and (6-3.16), the vector $x(\tau)$ does have a quasi periodicity as

$$x(\tau+T)=x(\tau)e^{j\omega T}. \qquad (6\text{-}3.19)$$

Then the vector $f(\tau)$ written as

$$f(\tau)=x(\tau)e^{-j\omega\tau} \qquad (6\text{-}3.20)$$

is periodic, since

$$\begin{aligned}f(\tau+T)&=x(\tau+T)e^{-j\omega(\tau+T)}\\ &=x(\tau)e^{-j\omega\tau}\\ &=x(\tau).\end{aligned} \qquad (6\text{-}3.21)$$

Then we obtain

$$\begin{aligned}x(\tau)&=f(\tau)e^{j\omega\tau}\\ &=e^{j\omega\tau}\sum_{m=-\infty}^{\infty} c_m e^{jm\frac{2\pi\tau}{T}},\end{aligned} \qquad (6\text{-}3.22)$$

where

$$\begin{aligned}c_m&=\frac{1}{T}\int_{nT}^{(n+1)T} f(\tau)e^{-jm\frac{2\pi\tau}{T}}\,d\tau\\ &=\frac{1}{T}\{\int_0^{t_1} f_1(\tau)e^{-jm\frac{2\pi\tau}{T}}\,d\tau+\int_0^{t_2} f_2(\tau)e^{-jm\frac{2\pi}{T}(\tau-t_1)}\,d\tau\}.\end{aligned}$$

If there exist integers p and q such that $pT=q\frac{2\pi}{\omega}$, the steady state solution is periodic with period pT. In general, however, the steady state solution is not periodic, but quasi periodic. If $u(\tau)$ is a constant function, the steady state solution is periodic with period T. For example consider the network shown

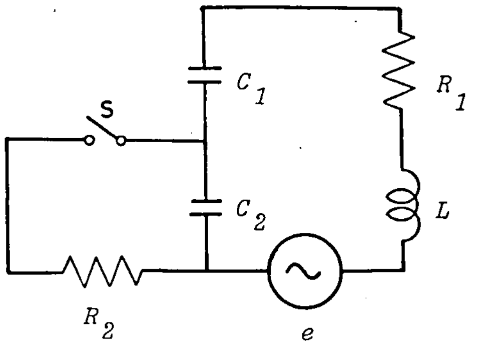


Fig.6-3.1 Example

in Fig.6-3.1, where in the first circuit mode the switch S is open, and in the second circuit mode S is closed. The period is T. If e is constant and $e=E$, the network has a periodic solution with period T. If $e=Ee^{\overline{j\omega t+\theta}}$, and if $mT=2n\pi/\omega$, then the network has a perodic solution with period mT, otherwise it has no periodic solution but a quasi periodic solution.

The following method of analizing such networks is simpler than the above. It is, however, unsuitable to study the problem on stability. Let x and u denote the state and the input vectors, respectively. In most cases, u is a solution of a linear differential equation. For example if $u=ae^{j\omega t}$, u is a solution of $\dot{u}=j\omega u$. Then we have a differential equation as

$$\begin{vmatrix} \dot{x} \\ \dot{u} \end{vmatrix} = \begin{vmatrix} A, & B \\ 0, & j\omega 1 \end{vmatrix} \begin{vmatrix} x \\ u \end{vmatrix}$$

$$\dot{y} = \widetilde{A}y, \qquad (6\text{-}3.23)$$

where $y = \begin{vmatrix} x \\ u \end{vmatrix}$, $\widetilde{A} = \begin{vmatrix} A, & B \\ 0, & j\omega 1 \end{vmatrix}$.

The Eq.(6-3.23) can be considered a state equation without input functions. The solution of Eq.(6-3.23) is

$$y(t)=e^{t\tilde{A}}y(0), \qquad (6\text{-}3.24)$$

where

$$e^{t\tilde{A}}=\begin{vmatrix} e^{tA}, & (A-j\omega 1)^{-1}(e^{tA}-e^{j\omega t}1)B \\ 0\ , & e^{j\omega t}1 \end{vmatrix}. \qquad (6\text{-}3.25)$$

Consider a network containing periodically operated switches with two circuit modes, then we obtain

$$y_{1,n}(t)=H_1(t)y_{2,n-1}(t_2) \qquad (6\text{-}3.26)$$

$$y_{2,n}(t)=H_2(t)y_{1,n}(t_1), \qquad (6\text{-}3.27)$$

which correspond to Eq.(6-3.9) and (6-3.10), respectively. Then we obtain

$$y_{1,n}(t_1)=D_1^{n-1}y_{2,1}(t_1)$$

$$y_{2,n}(t_2)=D_2^{n-1}y_{2,1}(t_2),$$

which correspond to Eq.(6-3.11) and (6-3.12), respectively. Let us define $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ as

$$\boldsymbol{D}_1 e^{j\omega nT}\triangleq\lim_{n\to\infty} D_1^n, \qquad \boldsymbol{D}_2 e^{j\omega nT}\triangleq\lim_{n\to\infty} D_2^n,$$

then we obtain the steady state solution as

$$\{y_{1,n}(t)\}_s=H_1(t)\boldsymbol{D}_2 y_{2,1}(t_2)e^{j\omega nT} \qquad (6\text{-}3.28)$$

$$\{y_{2,n}(t)\}_s=H_2(t)\boldsymbol{D}_1 y_{1,1}(t_1)e^{j\omega nT}, \qquad (6\text{-}3.29)$$

which correspond to Eq.(6-3.15) and (6-3.16), respectively.

This method is essentially the same as the previous method. This method, however, makes it evident that networks have periodic or quasi periodic solutions. The variables $y_{i,n}$ may be state variables of the i-th circuit mode, or may be voltages across all the capacitors and currents through all the inductors. .

6-4 EXPLICIT DEDUCTION OF THE INITIAL VALUES OF THE SECOND KIND FROM THOSE OF THE FIRST

6-4.1 THE DERIVATIVES IN THE MATRIX B

In Section 6-3, the analysis of the circuits is studied on the assumption that there exists no derivative term in B_i of Eq.(6-3.3) and (6-3.4). In general, however, there exist derivatives in B_i as mentioned in Chapter 2, 3, 4 and 5, then at $t=0$, there exist impulsive terms $(\frac{du}{dt}\big|_{t=0})$. Here it is shown that the matrix B without impulsive terms can be obtained by choosing initial values properly as follows.

In RLC networks there exists derivatives in B_i when loops containing only capacitor- and voltage-source-branches (capacitor- and voltage-source-only loops) are produced by closing *C*-switches, or when cut-sets containing only inductor- and current-source-branches (inductor- and current- source-only cut-sets) are produced by *L*-switches.

Consider the case where capacitor- and voltage-source-only loops are produced at $t=0$ by closing some *C*-switches. At $t=+0$, the Kirchhoff's voltage law in the loops states

$$[1, F_{SC}]\begin{vmatrix} v_S^{+0} \\ v_C^{+0} \end{vmatrix} = e_S, \qquad (6\text{-}4.1)$$

where the matrix F_{SC} is the same as is used in Chapter 2.

Let q. denote a vector of electric charges associated with the elements as noted by the subscripts. Then we obtain

$$[-F'_{SC}, 1]\begin{vmatrix} q_S^{+0} \\ q_C^{+0} \end{vmatrix} = 0.^{@} \qquad (6\text{-}4.2)$$

@ The case, where the right hand term is nonzero, is considred in Section 6-7.

Since

$$\begin{vmatrix} C_1, & 0 \\ 0 & , C_2 \end{vmatrix} \begin{vmatrix} v_S^{+0} \\ v_C^{+0} \end{vmatrix} = \begin{vmatrix} q_S^{+0} \\ q_C^{+0} \end{vmatrix}, \qquad (6\text{-}4.3)$$

the initial voltages of the second kind across the twig-capacitors are explicitly given as

$$v_C^{+0} = \mathcal{C}^{-1} F'_{SC} C_1 e_S. \qquad (6\text{-}4.4)$$

When inductor- and current-source-only cut-sets are produced by opening some L-switches, the initial currents of the second kind through the link-inductors are obtained as

$$i_L^{+0} = -\mathcal{L}^{-1} F_{L\Gamma} L_2 j_\Gamma. \qquad (6\text{-}4.5)$$

By choosing Eq.(6-4.4) and (6-4.5) as the initial values, the state equations without impulsive term in B_i can be obtained. (*i.e.* the terms $B\frac{du}{dt}\Big|_{t\geqq 0}$ can be replaced by the initial values and $B\frac{du}{dt}\Big|_{t>0}$)

In RLCT networks, the vectors v_C^{+0} and i_L^{+0} are given as

$$v_C^{+0} = \mathcal{C}^{-1} \widetilde{F}'_{SC} C_1 e_S, \qquad (6\text{-}4.6)$$

$$i_L^{+0} = -\mathcal{L}^{-1} \widetilde{F}_{L\Gamma} L_2 j_\Gamma. \qquad (6\text{-}4.7)$$

In RCG networks, the vector v_C^{+0} is given as

$$v_C^{+0} = \mathcal{C}^{-1} A'_{SC} C_1 e_{C1} - \mathcal{C}^{-1} A'_{SC} C_1 F_{C\Sigma} G_0^{-1} F'_{\Sigma\Sigma} R_1^{-1} e_{\Sigma 1} - \mathcal{C}^{-1} A'_{SC} C_1 F_{C\Sigma} G_0^{-1} j_{\Sigma 2}. \qquad (6\text{-}4.8)$$

In active networks, the initial values cannot be obtained explicitly as those in passive networks. If the matrix B in Eq.(5-4.9) is zero, the equalities corresponding to Eq.(6-4.1) and (6-4.2) can be obtained as

$$[-A, 1] \begin{vmatrix} v_C^{+0} \\ v_S^{+0} \end{vmatrix} + \widetilde{e}_S + \widetilde{j}_S = 0 \qquad (6\text{-}4.9)$$

$$[1,-F]\begin{vmatrix} q_C^{+0} \\ q_S^{+0} \end{vmatrix}=0, \tag{6-4.10}$$

where the matrices A and F are the same as are used in Chapter 5.

Then we obtain

$$v_C^{+0}=-(C_C-FC_SA)^{-1}F(e_S+j_S). \tag{6-4.11}$$

6-4.2 THE INITIAL VALUES OF THE FIRST AND THE SECOND KINDS

In Section 6-3, the networks are analized on the assumption that the initial values at a circuit mode can be obtained from the final values at the previous circuit mode, that is, the initial vales of the second kind can be obtained those of the first. The methods for obtaining the initial values of the second kind from those of the first were provided by S.Hayashi[28] and others[32]. However, they are not formulated. Here we derive a formula for RLC, RLCT and RCG networks.

RLC networks

When capacitor-only loops are produced by closing C-switches, the initial voltages of the second kind across the capacitors in the loops may be not equal to those of the first. Consider the network where the capacitor-only loops are produced at $t=0$ by closing C-switches. Kirchhoff's voltage law states

$$[1,F_{SC}]\begin{vmatrix} v_S^{+0} \\ v_C^{+0} \end{vmatrix}=0, \tag{6-4.11}$$

where the matrix F_{SC} is the same as is used in Chapter 2.
Let $q_\cdot^{-0}$ and $q_\cdot^{+0}$ denote the vectors of the initial electric charges of the first and the second kinds, respectively. From the law of conservation of electric charges, we obtain

$$[-F'_{SC},1]\begin{vmatrix} q_S^{+0} \\ q_C^{+0} \end{vmatrix}=[-F'_{SC},1]\begin{vmatrix} q_S^{-0} \\ q_C^{-0} \end{vmatrix}. \qquad (6\text{-}4.13)$$

The relations between the vectors of electric charges and voltages across capacitors are given as

$$C_1 v_S^{+0}=q_S^{+0}, \qquad C_2 v_C^{+0}=q_C^{+0} \qquad (6\text{-}4.14)$$
$$C_1 v_S^{-0}=q_S^{-0}, \qquad C_2 v_C^{-0}=q_C^{-0}$$

From Eq.(6-4.12), (6-4.13) and (6-4.14), the explicit relation between the initial voltages across the capacitors of the first and the second kinds are obtained as

$$v_C^{+0}=-\mathcal{C}^{-1}F'_{SC}C_1 v_S^{-0}+\mathcal{C}^{-1}C_2 v_C^{-0}. \qquad (6\text{-}4.15)$$

When inductor-only cut-sets are produced by opening L-switches, the initial currents of the second kind through inductors may not equal to those of the first. The initial currents of the second kind through inductors are derived from those of the first as

$$i_L^{+0}=\mathcal{L}^{-1}L_1 i_L^{-0}+\mathcal{L}^{-1}F_{L\Gamma}L_2 i_\Gamma^{-0}. \qquad (6\text{-}4.16)$$

RLCT networks

An RLCT network can be treated as an RLC network, as mentioned in Chapter 3, by regarding every matrix $\tilde{F}_{..}$ as the corresponding matrix $F_{..}$. Therefore the initial values of the second kind are obtained from those of the first as

$$v_C^{+0}=-\mathcal{C}^{-1}\tilde{F}'_{SC}C_1 v_S^{-0}+\mathcal{C}^{-1}C_2 v_C^{-0} \qquad (6\text{-}4.17)$$
$$i_L^{+0}=\mathcal{L}^{-1}L_1 i_L^{-0}+\mathcal{L}^{-1}\tilde{F}_{L\Gamma}L_2 i_\Gamma^{-0}. \qquad (6\text{-}4.18)$$

RCG network

An RCG network can be treated as an RLC network by regarding the matrix A_{SC} as F_{SC}, therefore the initial voltages of the second kind across the capacitors are

$$v_C^{+0}=-\mathcal{C}^{-1}A'_{SC}C_1 v_{C1}^{-0}+\mathcal{C}^{-1}C_2 v_{C2}^{-0}, \qquad (6\text{-}4.19)$$

where the matrices are those used in Chapter 4.

Active networks

The initial values of the second kind cannot explicitly be obtained from those of the first unless the matrix B in Eq.(5-4.9) is zero. Consider the case of B=0. From Eq.(5-4.9), we obtain

$$[-A,1]\begin{vmatrix} v_C^{+0} \\ v_S^{+0} \end{vmatrix}=0 \tag{6-4.20}$$

$$[1,-F]\begin{vmatrix} q_C^{+0} \\ q_S^{+0} \end{vmatrix}=[1,-F]\begin{vmatrix} q_C^{-0} \\ q_S^{-0} \end{vmatrix}, \tag{6-4.21}$$

which correspond to Eq.(6-4.12) and (6-4.13), repectively. Then we obtain

$$v_C^{+0}=\boldsymbol{C}^{-1}(C_C v_C^{-0}-FC_S v_S^{-0}), \tag{6-4.22}$$

where $\boldsymbol{C}=C_C-FC_SA$.

Consider the case of B≠0. In the process of deducing state equations from network-equations, Eq.(5-4.9) and (5-4.11), we obtain the eqalities,

$$\left.\begin{array}{ll} -A_2 v_C+v_{S2}=0 & (\text{ in Eq.(5-4.16)}) \\ -{}_1A_1 v_C+{}_2v_{S2}=0 & (\text{ in Eq.(5-4.26)}) \\ \vdots & \end{array}\right\} \tag{6-4.23}$$

if the voltage- and the current-sources are neglected.

Let Eq.(6-4.23) be represented by one matric equality as

$$\widetilde{A}v_C+v_S=0. \tag{6-4.24}$$

Then the initial voltages of the second kind are obtained, by replacing $\widetilde{A}$ with A in Eq.(6-4.22), as

$$v_C^{+0}=\boldsymbol{C}^{-1}(C_C v_C^{-0}-FC_S v_S^{-0}), \tag{6-4.25}$$

where $C=C_C-FC_S\tilde{A}$. Note that the rows of $\tilde{A}$ must be rearranged such that the link-capacitors corresponding to the columns of F are identiacal with those to the rows of A.

6-5 TWO KINDS OF INITIAL VALUES AND APPROXIMATE SOLUTIONS OF NETWORKS

6-5.1 INTRODUCTION

As mentioned in Section 6-4, the initial values of the second kind are not always equal to those of the first, that is, the voltage across a capacitor or the current through an inductor may jump instantaneously from an initial value of the first kind to that of the second. From a topological point of view, the above phenomenon arises when capacitor-only loops or capacitor- and independent voltage source-only loops"inductor-only cut-sets or inductor- and independent current source-only cut-sets" are produced by closing C-switches"open L-switches" in RLC networks. A condenser"coil" used in an actual network is equivalent to a capacitor"inductor" with a stray series-resistor of a small value and with a stray parallel-resitor of a large value. Then in its corresponding graph, there exists a normal tree containing all the capacitor-branches but no inductor-branch. In such a network containing periodically operated switches, the state variables for all the circuit modes are the same. The initial values in a circuit mode are always equal to the final values in the previous circuit mode, that is, the initial values of the second kind are always equal to those of the first. However in order to simplify the analysis, the above mentioned stray resistors connected in series and in parallel with capacitors and inductors are often neglected. In such networks, the initial values of the second kind are not always equal to those of the first. It is known that the stored energy in reactive elements decreases instantaneously at switch-operations.

Let us consider the networks N and N_e defined as

N: a network with stray series-resistor and with stray parallel-resistors

N_e: a network obtained from N by contracting the stray series-resistors and deleting the stray parallel-resistors.

Let A and A_e denote the A-matrices in the state equations for N and N_e, respectively. The matrix A has very large eigen values due to the stray series-resistors r with capacitors and the stray parallel-resistors g with inductors. The matrix A_e has no such eigen values. In general if a matrix A has eigen values which are much larger than the others, it is difficult to obtain the numerical solution of e^{tA} by the usual computations. On the contrary, it is not difficult to obtain that of e^{tA_e}. Then it is easier to obtain the solution for N_e than for N. It is shown in the following sections that the solution for N_e is an approximation for N.

In Section 6-5.2, the propositions for the analytical proof of the above point are posed.

In Section 6-5.3 and 4, the propositions in Section 6-5.2 are proved.

6-5.2 THE MATRICES A AND A_e

Let us pose the following propositions.

Proposition 6-5.1

The voltages across capacitors and the currents through inductors in N at time $t\simeq 0$ are nearly equal to those initial values of the second kind in N_e.

Proposition 6-5.2

The voltages across capacitors and the currents through inductors in N at time $t\gg 0$ are nearly equal to those in N_e.

It is evident that if the propositions are proved the solution for N_e is an approximation for N.

Let us derive the matrices A and A_e as described below. The fundamental loop matrix of graph G corresponding to N for a normal tree is written by

$$\begin{array}{c} \\ (r) \\ (R) \\ (L) \\ (\Gamma) \end{array}
\begin{array}{c} \begin{array}{cccccccc} (r) & (R) & (L) & (\Gamma) & (S) & (C) & (G) & (g) \end{array} \\ \left|\begin{array}{cccc|cccc} 1, & 0, & 0, & 0 & 1 & ,F_{r2}, & 0 & ,0 \\ 0, & 1, & 0, & 0 & 0 & ,F_{R2}, & F_{RG}, & 0 \\ 0, & 0, & 1, & 0 & 0 & ,F_{12}, & F_{LG}, & F_{1g} \\ 0, & 0, & 0, & 1 & 0 & ,0 & ,0 & ,-1 \end{array}\right| \end{array}, \qquad (6\text{-}5.1)$$

where the first rows correspond to stray link-resistor-branches r in series with capacitor-branches S. The last columns correspond to stray twig-resistor-branches g in parallel with inducror-branches Γ.

Consider the graph G_e corresponding to N_e. The graph G_e is obtained from G by contracting the stray series resistor-branches and deleting the stray parallel ones. The fundamental loop matrix of G_e may be written by

$$\begin{vmatrix} 1,0,0,F_{SC}, & 0 \;, & 0 \\ 0,1,0,F_{RC}, & F_{RG}, & 0 \\ 0,0,1,F_{LC}, & F_{LG}, & F_{L\Gamma} \end{vmatrix} . \qquad (6\text{-}5.2)$$

Since there exists a normal tree such that S are links and Γ twigs, the relation (6-5.3) for the submatrices (6-5.1) and (6-5.2) can be established.

$$\begin{aligned} &F_{r2}=F_{SC}, \quad F_{R2}=F_{RC}, \quad F_{12}=F_{LC}, \\ &F_{1G}=F_{LG}, \quad F_{1g}=F_{L\Gamma}. \end{aligned} \qquad (6\text{-}5.3)$$

The matrix A_e is given as

$$A_e=\begin{vmatrix} C^{-1}, & 0 \\ 0 \;, & L^{-1} \end{vmatrix}\begin{vmatrix} -y \;, & H \\ -H', & -z \end{vmatrix}. \qquad (6\text{-}5.4)$$

The matrices on the right hand side of (6-5.4) are the same as are used in Chapter 2.

Let $r''g''$ denote the resistance matrix of the resistor-branches $r''$the conductance matrix of the resistor-branches g''. Then the matrix A may be written as

$$A=\begin{vmatrix} C_1^{-1}, & 0 \;, & 0 \;, & 0 \\ 0 \;, & C_2^{-1}, & 0 \;, & 0 \\ 0 \;, & 0 \;, & L_1^{-1}, & 0 \\ 0 \;, & 0 \;, & 0 \;, & L_2^{-1} \end{vmatrix}\begin{vmatrix} -r^{-1} \;, & -r^{-1}F_{r2} \;, & 0 \;, & 0 \\ -F_{12}'r^{-1}, & -(y+F_{r2}'r^{-1}F_{r2}), & H \;, & 0 \\ 0 \;, & -H' & ,-(z+F_{1g}g^{-1}F_{1g}'), & F_{1g}g^{-1} \\ 0 \;, & 0 \;, & g^{-1}F_{1g} \;, & -g^{-1} \end{vmatrix}, \qquad (6\text{-}5.5).$$

where C_1 and C_2"L_1 and L_2" are the capacitance matrices"the inductance matrices" for the link- and the twig-capacitors"inductors", respectively, in N_e. The left upper and the right lower parts of A are of large values.

6-5.3 THE PROOF OF PROPOSITON 6-5.1

Let us give the proof of Proposition 6-5.1 in Section 6-5.2.

The state equation of N is given by

$$\dot{x}(t)=Ax(t)+Bu(t), \tag{6-5.6}$$

where $x(t)$ is the vector whose elements are the voltages across the capacitors and the currents through the inductors.

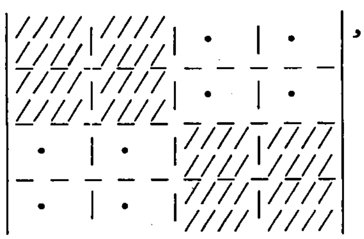
The solution of Eq.(6-5.6) may be given by

$$x(t)=e^{tA}x(+0)+\int_0^t e^{(t-\tau)A}Bu(\tau)d\tau. \tag{6-5.7}$$

The matrix e^{tA} is given by a form of an infinite series as

$$e^{tA}=\sum_{n=0}^{\infty}\frac{1}{n!}(tA)^n. \tag{6-5.8}$$

The matrix e^{tA} at $t\simeq 0$ is nearly equal to $1+tA$. This matrix has a form of



(6-5.9)

where the shaded portions are the submatrices for the stray resistors r and g.

The submatrices not corresponding to the stray resistors in (6-5.1) are approximated by zero matrices. Then we obtain

$$i_C-F'_{r2}i_r\simeq 0, \tag{6-5.10}$$

where $i_.$ is the current vector associated with the elements as denoted by the subscripts.

From (6-5.1), we obtain

$$i_S - i_r = 0. \qquad (6\text{-}5.11)$$

From Eq.(6-5.10) and (6-5.11), we obtain

$$i_C - F'_{r2} i_S \simeq 0, \quad (\text{for } t \gtrsim 0). \qquad (6\text{-}5.12)$$

Integrating Eq.(6-5.12) from $t=-0$ to $t=+0$, we obtain the following equation for the electric charges;

$$q_C^{-0} - F'_{r2} q_S^{-0} \simeq q_C^{+0} - F'_{r2} q_S^{+0}. \qquad (6\text{-}5.13)$$

From Eq.(6-5.3), we see that Eq.(6-5.13) is identical with Eq.(6-4.13). Then the voltages across the capacitors at $t \simeq 0$ in N are nearly equal to the initial voltages of the second kind across the capacitors in N_e.

Similarly it can be proved that the currents through the inductors at $t \simeq 0$ in N are nearly equal to the initial currents of the second kind through the inductors in N_e.

Consequently a proof is given to Proposition 6-5.1.

6-5.4 A PROOF OF PROPOSITION 6-5.2

Let us define the matrix P as

$$P = \begin{vmatrix} (1+F_{r2}F'_{r2})^{-1} & ,(1+F_{r2}F'_{r2})^{-1}F_{r2}, & 0 & , & 0 \\ -(1+F'_{r2}F_{r2})^{-1}F'_{r2}, & (1+F'_{r2}F_{r2})^{-1} & , & 0 & , & 0 \\ 0 & , & 0 & , & (1+F_{1g}F'_{1g})^{-1} & ,(1+F_{1g}F'_{1g})^{-1}F_{1g} \\ 0 & , & 0 & ,-(1+F'_{1g}F_{1g})^{-1}F_{1g}, & (1+F'_{1g}F_{1g})^{-1} \end{vmatrix} \qquad (6\text{-}5.14)$$

The inverse matrix of P is given as

$$P^{-1}=\left|\begin{array}{cccc} 1, & -F_{r2}, & 0, & 0 \\ F'_{r2}, & 1, & 0, & 0 \\ 0, & 0, & 1, & -F_{1g} \\ 0, & 0, & F'_{1g}, & 1 \end{array}\right|. \tag{6-5.15}$$

The matrix A, (6-5.5) is similar to the following matrix,

$$PAP^{-1}=\left|\begin{array}{cccc}
-(1+F_{r2}F'_{r2})^{-1}\{(C_1^{-1}+F_{r2}C_2^{-1}F'_{r2})\boldsymbol{r}^{-1}(1+F_{r2}F'_{r2})+\underline{F_{r2}C_2^{-1}\boldsymbol{y}F'_{r2}}\}, & -(1+F_{r2}F'_{r2})^{-1}F_{r2}C_2^{-1}, & (1+F_{r2}F'_{r2})^{-1}F_{r2}C_2^{-1}H, & -(1+F_{r2}F'_{r2})^{-1}F_{r2}C_2^{-1}HF_{1g} \\
(1+F'_{r2}F_{r2})^{-1}\{(F'_{r2}C_1^{-1}-C_2^{-1}F'_{r2})\boldsymbol{r}^{-1}(1+F_{r2}F'_{r2})-\underline{C_2^{-1}\boldsymbol{y}F'_{r2}}\}, & -(1+F'_{r2}F_{r2})^{-1}C_2^{-1}\boldsymbol{y}, & (1+F'_{r2}F_{r2})^{-1}C_2^{-1}H, & -(1+F'_{r2}F_{r2})^{-1}C_2^{-1}HF_{1g} \\
-(1+F_{1g}F'_{1g})^{-1}L_1^{-1}H'F'_{r2}, & -(1+F_{1g}F'_{1g})^{-1}L_1^{-1}H', & -(1+F'_{1g}F_{1g})^{-1}L_1^{-1}\boldsymbol{z}, & (1+F_{1g}F'_{1g})^{-1}\{(L_1^{-1}F_{1g}-F'_{1g}L_2^{-1})\boldsymbol{g}^{-1}(1+F'_{1g}F_{1g})+\underline{L_1^{-1}\boldsymbol{z}F_{1g}}\} \\
(1+F'_{1g}F_{1g})^{-1}F'_{1g}L_1^{-1}H'F'_{r2}, & (1+F'_{1g}F_{1g})^{-1}F_{1g}L_1^{-1}H', & (1+F'_{1g}F_{1g})^{-1}F'_{1g}L_1^{-1}\boldsymbol{z}, & -(1+F'_{1g}F_{1g})^{-1}\{(L_2^{-1}+F'_{1g}L_1^{-1}F_{1g})\boldsymbol{g}^{-1}(1+F'_{1g}F_{1g})+\underline{F_{1g}L_1^{-1}\boldsymbol{z}F_{1g}}\}
\end{array}\right|. \tag{6-5.6}$$

Let $\widetilde{A}$ denote the matrix obtained by omitting the underlined terms in (6-5.6). The matrix $\widetilde{A}$ is nearly equal to PAP^{-1} since the values of $\boldsymbol{r}^{-1}$ and $\boldsymbol{g}^{-1}$ are much larger than those of the other terms in PAP^{-1}. The matrix $Q\widetilde{A}Q^{-1}$ in (6-5.17) is similar to $\widetilde{A}$.

$$Q\widetilde{A}Q^{-1}=\left|\begin{array}{cccc} A_{11}, & A_{12}, & A_{13}, & A_{14} \\ A_{21}, & A_{22}, & A_{23}, & A_{24} \\ A_{31}, & A_{32}, & A_{33}, & A_{34} \\ A_{41}, & A_{42}, & A_{43}, & A_{44} \end{array}\right|, \tag{6-5.17}$$

where A_{ij} are shown in Appendix VII and the matrix Q, which is determined by the network-topology and network-element values, is

$$Q=\begin{vmatrix} 1, & 0, & 0, & 0 \\ (1+F'_{r2}F_{r2})^{-1}(F'_{r2}C_1^{-1}-C_2^{-1}F'_{r2})(C_1^{-1}+F_{r2}C_2^{-1}F'_{r2})^{-1}(1+F_{r2}F'_{r2}), & 1, & 0, & 0 \\ 0, & 0, & 1, & (1+F_{1g}F'_{1g})^{-1}(L_1^{-1}F_{1g}-F_{1g}L_2^{-1})(L_1^{-1}+F_{1g}L_2^{-1}F'_{1g})^{-1}(1+F'_{1g}F_{1g}) \\ 0, & 0, & 0, & 1 \end{vmatrix} \tag{6-5.18}$$

The submatrices A_{ij}, except A_{11} and A_{44}, do not contain r^{-1} and g^{-1}. Therefore the matrix QAQ^{-1} is nearly equal to a matrix such as

$$QAQ^{-1}=\begin{vmatrix} A_{11}, & 0, & 0, & 0 \\ 0, & A_{22}, & A_{23}, & 0 \\ 0, & A_{32}, & A_{33}, & 0 \\ 0, & 0, & 0, & A_{44} \end{vmatrix}. \tag{6-5.19}$$

Then the eigen values of A are nearly equal to those of A_{11}, A_{44} and $\begin{vmatrix} A_{22}, & A_{23} \\ A_{32}, & A_{33} \end{vmatrix}$. The eigen values of A_{11} and A_{44} are very large.

Now we show the following equality,

$$\begin{vmatrix} A_{22}, & A_{23} \\ A_{32}, & A_{33} \end{vmatrix} = \begin{vmatrix} C^{-1}, & 0 \\ 0, & L^{-1} \end{vmatrix} \begin{vmatrix} -y, & H \\ -H, & -z \end{vmatrix}. \tag{6-5.20}$$

From Lemma A-3-3, we obtain

$$(C_1^{-1}+F_{r2}C_2^{-1}F'_{r2})^{-1}=C_1-C_1F_{r2}(C_2+F'_{r2}C_1F_{r2})^{-1}F'_{r2}C_1. \tag{6-5.21}$$

Eq. (6-5.3) leads to

$$(C_1^{-1}+F_{r2}C_2^{-1}F'_{r2})^{-1}=C_1-C_1F_{r2}C^{-1}F'_{r2}C_1. \qquad (6\text{-}5.22)$$

Similarly we obtain

$$(L_2^{-1}+F'_{1g}L_1^{-1}F_{1g})^{-1}=L_2-L_2F'_{1g}L^{-1}F_{1g}L_2. \qquad (6\text{-}5.23)$$

From Eq.(6-5.22) and (6-5.23), we obtain

$$\begin{aligned}&(1+F'_{r2}F_{r2})^{-1}\{(F'_{r2}C_1^{-1}-C_2^{-1}F'_{r2})(C_1^{-1}+F_{r2}C_1^{-1}F'_{r2})^{-1}F_{r2}C_2^{-1}+C_2^{-1}\}\\ &=(C_2+F'_{SC}C_1F_{SC})^{-1}\\ &=C^{-1}, \end{aligned} \qquad (6\text{-}5.24)$$

$$\begin{aligned}&(1+F'_{1g}F_{1g})^{-1}\{(L_1^{-1}F_{1g}-F_{1g}L_2^{-1})(L_2^{-1}+F'_{1g}L_1^{-1}F_{1g})^{-1}F'_{1g}L_1^{-1}-L_1^{-1}\}\\ &=(L_1+F_{L\Gamma}L_2F'_{L\Gamma})^{-1}\\ &=L^{-1}. \end{aligned} \qquad (6\text{-}5.25)$$

Therefore Eq.(6-5.20) holds. Then this completes the proof of Proposition 6-5.2.

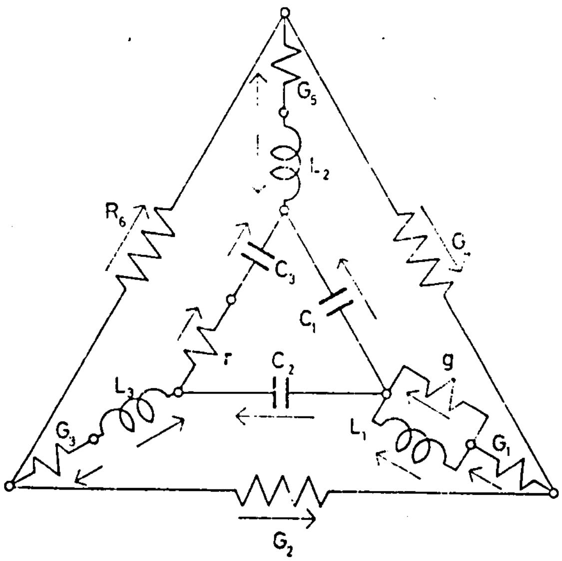
6-5.4 EXAMPLE

Consider a network N shown in Fig.6-5.1. Opening the resistor g, and shorting the resistor r, we obtain a network N_e. There is no capacitor-only loop or inductor-only cut-set in G corresponding to N. There are a capacitor-only loop and an inductor-only cut-set in G_e corresponding to N_e.

The eigen values of A_e and A for several values of the resistor r and g are shown in Table 6-5.1. Those eigen values are obtained by the method of Danilevsky.

The voltage across C_2 and the current through L_1 for time t are computed for the initial values shown in Table 6-5.2. The results are shown in Fig.6-5.2 - 6-5.5.

Table 6-5.1 shows that, in Case (1) and (2), the eigen values of A except those with large negative values are nearly equal to



$C_1 = 1.0\ F$ $\quad L_1 = 2.0\ H$

$C_2 = 2.0\ F$ $\quad L_2 = 0.5\ H$

$C_3 = 3.0\ F$ $\quad L_3 = 1.0\ H$

$G_1 = 0.5$ mho $\quad G_2 = 3.0$ mho

$G_3 = 0.2$ mho $\quad G_4 = 0.3$ mho

$G_5 = 2.5$ mho $\quad R_6 = 3.0$ ohm

Fig. 6-5.1 Example

Table 6-5.1 Eigen values of A_e and A

N_e	Case (1)	Case (2)	Case (3)	Case (4)
-0.1216325	-0.1216239	-0.1215665	-0.1213041	-0.1209806
-4.692572	-4.692756	-4.694653	-4.707070	-4.733613
-0.03248342	-0.03248453	-0.03246332	-0.03238303	-0.03228288
-1.400931	-1.397356	-1.365844	-1.239297	-1.107467
	-1833.333	-183.3304	-36.64787	-18.28363
	-3509.223	-359.2551	-79.38541	-44.52202
r (ohm)	1×10^{-3}	1×10^{-2}	5×10^{-2}	1×10^{-1}
g (mho)	1×10^{-3}	1×10^{-2}	5×10^{-2}	1×10^{-1}

Table 6-5.2 Initial values

	V_{C_1}	V_{C_2}	V_{C_3}	I_{L_1}	I_{L_2}	I_{L_3}
$t=-0$	-5.0 V	0.0 V	10.0 V	0.0 A	10.0 A	2.0 V
$t=+0$	35/11 V	-45/11 V	80/11 V	-12/7 A	22/ 7 A	-10/7 A

The values for $t=+0$ are computed from those for $t=-0$.
They are the initial values of the second kind in N_e.

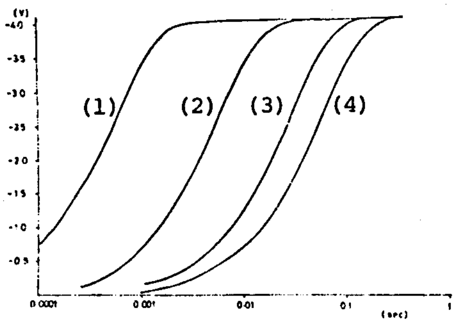


Fig.6-5.2 Voltage across C_2
$(t<1)$

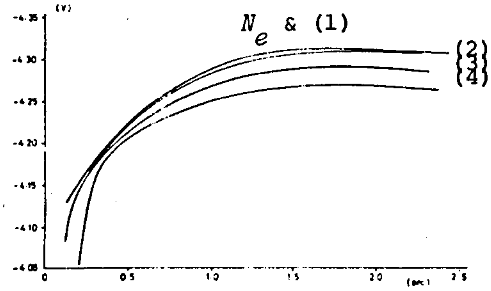


Fig.6-5.3 Voltage across C_2
$(t<2.5)$

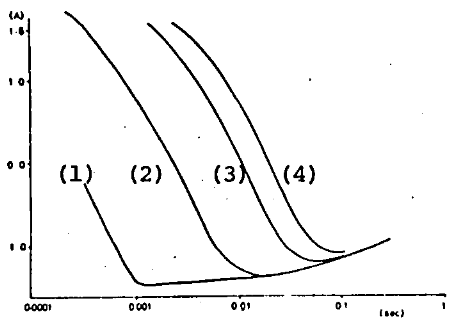


Fig.6-5.4 Current through L_3
$(t<1)$

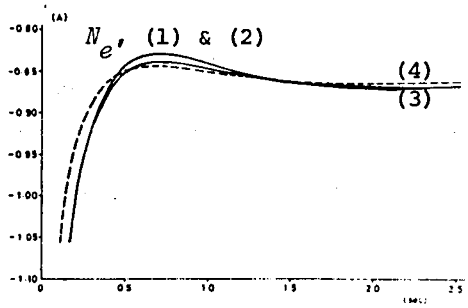


Fig.6-5.5 Current through L_3
$(t<2.5)$

those of A_e. The voltages and the currents in N agree precisely with those in N_e except small t, as shown in Fig.6-5.2 - 6-5.5. In case (3) and (4), however, the eigen values of A and those of A_e do not agree. The voltages and the currents in N are different from those in N_e.

From the above discussions, it is concluded that if r and g are small, the voltages and the currents in the network N_e are nearly equal to those in N for all t except small t. The voltages and the currents in N approach from the initial values of the first kind to those of the second kind in N_e in a very short period of time. After that time, the network N_e gives an approximate solution to N.

6-6 STABILITY

6-6.1 INTRODUCTION

In Section 6-3, on the assumption that $D_i^n \to 0$ as $n \to \infty$, steady state solutions of linear networks containing periodically operated switches are obtained. Whether $D_i^n \to 0$ or not as $n \to \infty$, is called the problem on stability. Only S.Hayashi[28] has discussed this problem. His approach of this problem, however, is the examination of the eigen values of D_i. Here without examining eigenvalues, we get the conclution that passive linear networks containing periodically operated switches always have steady state solutions.

6-6.2 PROPERTIES OF D_i

Property 6-6.1

Every eigen value of D_i is equal to that of D_j $\{i \neq j\}$.

Proof: From Section 6-3, the matrices D_i and D_{i+1} are:

$$D_i = H_i(t_i)H_{i+1}(t_{i+1})\cdots\cdots H_{i-1}(t_{i-1}) \tag{6-6.1}$$

$$D_{i+1} = H_{i+1}(t_{i+1})\cdots\cdots H_i(t_i). \tag{6-6.2}$$

Then the matrices D_i and D_{i+1} are written as

$$D_i = H_i(t_i)\ddot{H}_i$$

$$D_{i+1} = \ddot{H}_i H_i(t_i),$$

where $\ddot{H}_i = H_{i+1}(t_{i+1})\cdots\cdots H_n(t_n)H_1(t_1)\cdots\cdots H_{i-1}(t_{i-1})$.

From Lemma A-3.4, the characteristic equation for D_i is the same as that for D_{i+1}. Then the eigen values of D_i are equal to those of D_{i+1}. Similarly the eigen values of D_{i+1} are equal to those of D_{i+2}. By induction, the eigen values of D_i are equal to those of D_j $\{j=1,\cdots,n.\ j\neq i\}$.

6-6.3 STABILITY OF A PASSIVE NETWORK SATISFYING ASSUMPTION 6-6.1

Now let us consider a passive network satisfying:

Assumption 6-6.1

A set of state variables for one circuit mode is identical with that of every other circuit mode.

It can be assumed that such a network has two circuit mode. The state equations of the first and the second circuit modes are, respectively,

$$\dot{x}=A_1x+B_1u_1, \tag{6-6.5}$$

$$\dot{x}=A_2x+B_2u_2. \tag{6-6.6}$$

From Assumption 6-6.1, the matrices D_1 and D_2 are written as

$$D_1=e^{t_1A_1}e^{t_2A_2}, \qquad D_2=e^{t_2A_2}e^{t_1A_1}, \tag{6-6.7}$$

where t_1 and t_2 are periods of the first and the second circuit modes, respectively.

Since the network considered is passive, the real parts of all eigen values of the matrices A_1 and A_2 are nonpositive.

In order to examine whether $D_i^n\to 0$ as $n\to\infty$, consider matric differential equations defined by

$$\dot{x}=A_ix \qquad \{i=1,2\}. \tag{6-6.8}$$

The solutions of Eq.(6-6.8) are

$$\dot{x}(t)=e^{tA_1}x(0) \qquad \{i=1,2\}. \tag{6-6.9}$$

Let us define a function;

$$\Delta \equiv x'(0)x(0)-x'(t)x(t). \tag{6-6.10}$$

From Eq. (6-6.9), we get

$$\Delta = x'(0)\{1-(e^{tA_i})'e^{tA_i}\}x(0). \tag{6-6.11}$$

For a linear passive network, the matrix $1-(e^{tA_i})'e^{tA_i}$ is positive semi-definite or positive definite.

Assume that the matrix $1-(e^{t_iA_i})'e^{t_iA_i}$ is positive definite. From Lemma A-3.5, there exists a real nonsingular matrix P_i such that

$$P_iP_i'=1-(e^{t_iA_i})'e^{t_iA_i}. \tag{6-6.12}$$

Consider a function defined by

$$x(t+T)=e^{t_1A_1}e^{t_2A_2}x(t), \tag{6-6.13}$$

where $T=t_1+t_2$,

and a function defined by

$$\tilde{\Delta} \equiv x'(t+T)x(t+T)-x'(t)x(t). \tag{6-6.14}$$

If Δ is negative, a system descrived by (6-6.13) is asympotic stable[34].

Substituting Eq. (6-6.13) into Eq. (6-6.14), we obtain

$$\tilde{\Delta}=x'(t)\{(e^{t_2A_2})'(e^{t_1A_1})'e^{t_1A_1}e^{t_2A_2}-1\}x(t). \tag{6-6.15}$$

From the matrix (6-6.12), Eq. (6-6.15) becomes

$$\tilde{\Delta}=-x'(t)\{P_2'P_2+Q'Q\}x(t), \tag{6-6.16}$$

where $Q=P_1e^{t_2A_2}$.

Since the matrices $P_2'P_2$ and $Q'Q$ are positive definite, Δ is negative.

If the matrix $1-(e^{tA_i})'e^{tA_i}$ is positive semi-definite, then, from Property 2-3.4 in Chapter 2, the graph G_i (corresponding to the i-th circuit mode) contains capacitor-only cut-set and/or inductor-only loop, or the graph G_i may have connected-components containing only capacitor- and/or inductor-branches.

If both the matrices $1-(e^{tA_i})'e^{tA_i}$ are positive semi-definite, there exist capacitor-only cut-sets and/or inductor-only loops in both G_1 and G_2. Note that we have assumed that both graphs G_1 and G_2 cannot have a non-separable component containing capacitor- and/or inductor-branches only.

In case when there exists a capacitor-only cut-set or an inductor-only loop in both G_1 and G_2, G_1 and G_2 are converted to equivalent graphs containing no capacitor-only cut-set and no inductor-only loop by the star-delta transformation as shown in Fig.6-6.1. Then the matrices $1-(e^{tA_i})'e^{tA_i}$ are positive definite.

When at least one of the matrices $1-(e^{tA_i})'e^{tA_i}$ is positive semi-definite, it can be assumed without loss of generality that the matrix $1-(e^{t_2A_2})'e^{t_2A_2}$ is positive definite. Then there exists a real nonsingular matrix P_2 such that

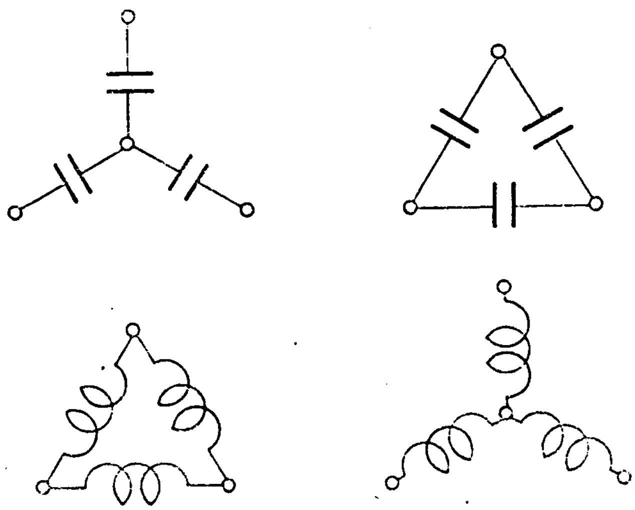


Fig.6-6.1

Example of star-delta transformation

$$P_2P_2' = 1-(e^{t_2A_2})'e^{t_2A_2}. \qquad (6\text{-}6.17)$$

From Lemma A-3.6, we obtain a real matrix P_1 such that

$$P_1P_1' = 1-(e^{t_1A_1})'e^{t_1A_1}. \qquad (6\text{-}6.18)$$

In Eq.(6-6.16), the matrix Q may singular. Since $P_2'P_2$ is positive definite, however, $\tilde{\Delta}$ is negative.

Consequently we obtain

Theorem 6-6.1

The linear passive networks containing periodically operated switches which satisfy Assumption 6-6.1 possess steady state solutions.

6-6.4 STABILITY OF LINEAR PASSIVE NETWORKS CONTAINING PERIODICALLY OPERATED SWITCHES

In Section 6-6.3 is studied the stability of linear passive networks containing periodically operated switches which satisfy Assumption 6-6.1. Now let us consider the stability of the networks which do not satisfy Assumption 6-6.1. In such networks, the final values of a cicuit mode may not be always equal to the initial values of the following circuit mode. If the total energy stored in all the reactive elements does not increase at the switch-operations, the network is stable since the network considered is passive. Let us considered the energy variations at the switch-operations.

As mentioned in Section 6-4, when *C*-switches are closed"*L*-switches are opened", the voltages across the capacitors"the currents through inductors" may vary.

Since the stability does not depend on the B-matrix of the state equations, networks where the voltage sources are contracted and the current sources are deleted in the original networks are considered here.

Let us consider the difference of the energy stored in capacitors when C-switches are closed. Let P_0 and P_N denote the energy stored just before and just after closing C-switches, respectively. Then we obtain

$$P_0=\frac{1}{2}(v'_{1,0}C_1v_{1,0}+v'_{2,0}C_2v_{2,0}) \tag{6-6.19}$$

$$P_N=\frac{1}{2}(v'_{1,N}C_1v_{1,N}+v'_{2,N}C_2v_{2,N}), \tag{6-6.20}$$

where the subscripts 1 and 2 denote link- and twig-capacitors, respectively.

A vector v_0 in case of an RLC network is defined as

$$v_0\triangleq[1,F_{SC}]\begin{vmatrix} v_{1,0} \\ v_{2,0} \end{vmatrix}, \tag{6-6.21}$$

where $[1,F_{SC}]$ is the matrix used in Chapter 2.

From (6-6.21), the difference of the energy stored in the capacitors is written as

$$P_0-P_N=\frac{1}{2}v'_0(C_1-C_1F_{SC}C^{-1}F'_{SC}C_1)v_0. \tag{6-6.22}$$

From Lemma A-3.3, Eq.(6-6.22) becomes

$$P_0-P_N=\frac{1}{2}v'_0(C_1^{-1}+F_{SC}C_2^{-1}F'_{SC})v_0. \tag{6-6.23}$$

Since the matrix $C_1^{-1}+F_{SC}C_2^{-1}F'_{SC}$ is positive definite, the total energy stored in the capacitors does not increase at closing C-switches.

Similarly it is proved that the total energy stored in the inductors does not increase at opening L-switches.

Consequently linear RLC networks containing periodically operated switches have steady state solutions.

In case of linear RLCT networks containing periodically operated switches, it can be proved, by replacing F_{SC} by $\widetilde{F}_{SC}$ in Eq.(6-6.21)-(6-6.23), that they have steady state solutions.

In case of linear RCG networks containing periodically operated switches, it can be proved, by replacing F_{SC} by A_{SC} in Eq.(6-6.21)-(6-6.23), that they have steady state solutions.

Consequently we obtain

Theorem 6-6.2

Linear passive networks containing periodically operated switches have steady state solutions.

6-6.5 STABILITY OF ACTIVE NETWORKS CONTAINING PERIODICALLY OPERATED SWITCHES

In the preceding sections, is studied stability of linear networks containing periodically operated switches where all the circuit modes are stable. Here we considered linear networks containing periodically operated switches, where the A-matrices of state equations for some circuit modes have eigen values whose real parts are positive and those for the other modes have no such eigen values. It is examined whether such networks are stable or not.

For this consideration, it is convenient to derive the Jordan canonical forms of the matrix D_i as

$$D_i = T^{-1} \begin{vmatrix} J_1, & 0, & \cdots & 0 \\ 0, & J_2, & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_s \end{vmatrix} T, \tag{6-6.24}$$

where J_k is a Jordan block as

$$J_k = \begin{vmatrix} \lambda_k, 1 & & & 0 \\ 0, & \lambda_k, 1 & & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{vmatrix}. \tag{6-6.25}$$

The dimension of the block J_k is assumed to be d_k, and λ_k is an eigen value of D_i. From Eq.(6-6.24), D_i^n is written as

$$D_i^n = T^{-1} \begin{vmatrix} J_1^n, & & & 0 \\ & J_2^n & & \\ & & \ddots & \\ 0 & & & J_s^n \end{vmatrix} T, \tag{6-6.26}$$

where

$$J_k^n = \begin{vmatrix} \lambda_k^n, n\lambda_k^{n-1}, \cdots, & \binom{n}{n-d_k+1} \lambda_k^{n-d_k+1} \\ \lambda_k^n, \quad \cdots & \vdots \\ \ddots & \vdots \\ 0 & \lambda_k^n \end{vmatrix}. \tag{6-6.27}$$

From Eq.(6-6.24)-(6-6.27), it is concluded that

1) If the absolute value of λ_k is less than 1 for all k, then $J_k^n \to 0$ as $n \to \infty$, and $D_k^n \to 0$ as $n \to \infty$.

2) If there exists at least one eigen value such that $|\lambda_k| > 1$, J_k^n does not converge. Then the network is unstable.

3) If there exist some eigen values λ_l such that $|\lambda_l|=1$, and if all other absolute values of $|\lambda_k|$ for $k\neq l$ are less than 1, we must consider two cases as follows;

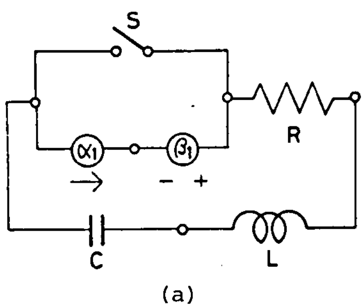
a) If $d_l=1$ for all l, the matrix D_i does not diverge.

b) If $d_l\neq 1$ for some l, the matrix D_i diverges. Then the network is unstable.

The stability of such networks is a complex problem. Consider the stability of a simple example shown in Fig.6-6.2.

Let A_1 and A_2 denote the A-matrices of the state equations for the first and the second circuit modes, respectively. In the state equations, the state variables are the voltage across the capacitor and the current through the inductor. The matrices A_1 and A_2 are written, respectively, as

$$A = \begin{vmatrix} 0 \;, 1/C \\ -1/L, R/L \end{vmatrix} \qquad A = \begin{vmatrix} 0 \;, \; 1/C \\ -1/L, -R/L \end{vmatrix}. \tag{6-6.28}$$



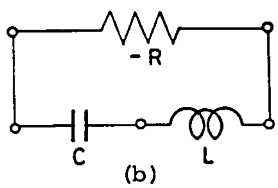
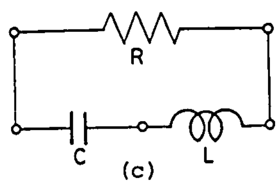


Fig.6-6.2 Example

(a) Original network, where $v_{\alpha 1} = 2Ri_{\alpha 1}$.

(b) The first circuit mode.

(c) The second circuit mode.



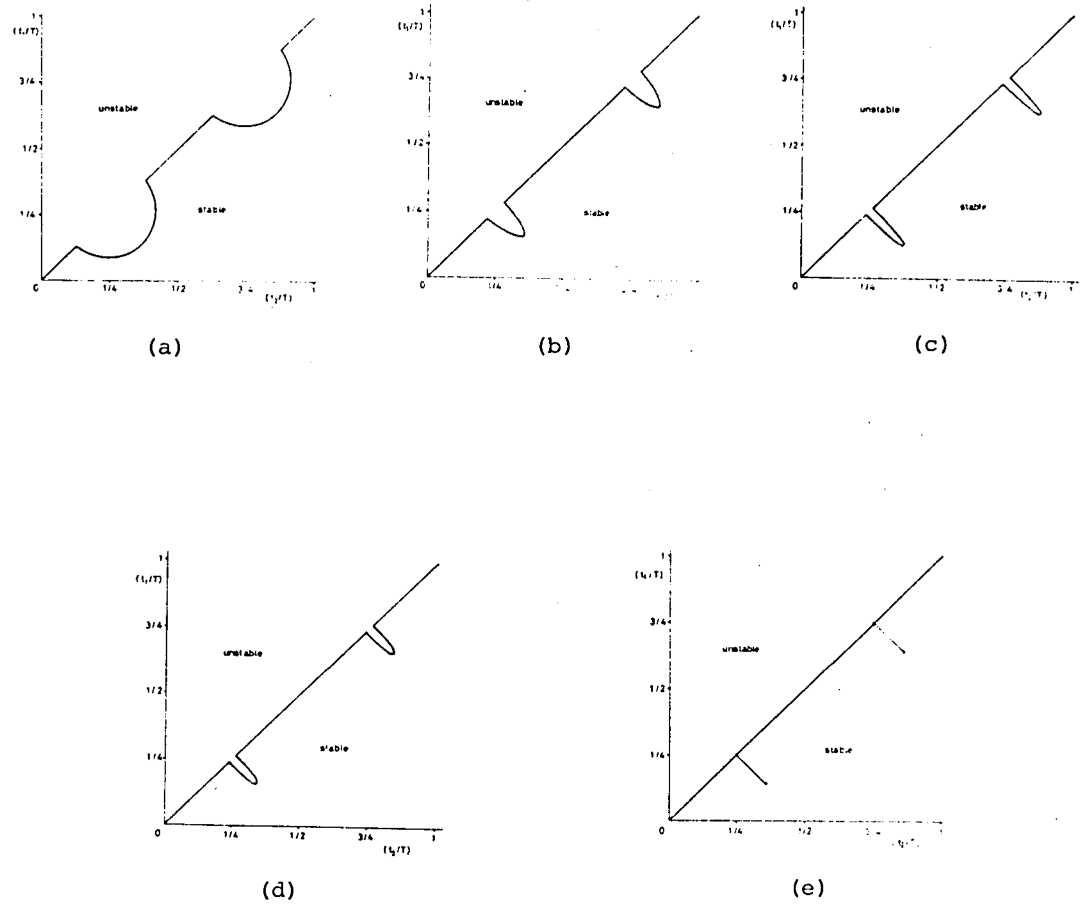


Fig.6-6.3 Stable regions of the network shown in Fig.6-6.2

	L (H)	C (F)	R (Ω)	τ	β	T
(a)	0.01	0.05	2.0	0.01	100	0.0628
(b)	0.005	0.001	0.5	0.02	444.4	0.014
(c)	0.01	0.005	0.2	0.1	141	0.044
(d)	0.005	0.001	0.3	0.033	446.2	0.014
(e)	0.005	0.005	0.2	0.1	199.7	0.031

The network for the first circuit mode has no steady state solution, but that for the second one has a steady state solution.

The eigen values of A_1 are

$$\lambda_1 = \frac{1}{\tau} \pm j\beta, \tag{6-6.29}$$

where $\frac{1}{\tau} = R/2L$, $\quad = \sqrt{1/LC - 1/\tau^2}$.

The eigen values of A_2 are

$$\lambda_2 = \frac{1}{\tau} \pm j\beta. \tag{6-6.30}$$

The graphs in Fig.6-6.3 show the stable regions in several values of the network-elements. It may be expected that the network will be stable for $t_1 < t_2$ since the networks for the two circuit modes are the same except the sign of the resistance. But from Fig.6-6.3, we see the network is not always stable for $t_1 < t_2$.

We investigate the stable regions in Fig.6-6.3 in detail. The matrix D_1 is written as

$$\begin{aligned} D_1 &= e^{t_1 A_1} e^{t_2 A_2} \\ &\triangleq e^{(t_1 - t_2)/\tau} \begin{vmatrix} r_{11}, r_{12} \\ r_{21}, r_{22} \end{vmatrix} \\ &\triangleq a \begin{vmatrix} r_{11}, r_{12} \\ r_{21}, r_{22} \end{vmatrix}, \end{aligned} \tag{6-6.31}$$

where

$$a = e^{(t_1 - t_2)/\tau}$$

$$r_{11} = \cos\beta t_1 \cos\beta t_2 - \frac{1}{}(\sin\beta t_1 \cos\beta t_2 - \cos\beta t_1 \sin\beta t_2) - (1 + \frac{2}{\beta^2 \tau^2}) \sin\beta t_1 \times \sin\beta t_2$$

$$r_{12} = \frac{1}{C\beta} \cos\beta t_1 \sin\beta t_2 - \frac{1}{C\beta^2 \tau} \sin\ t_1 \sin\ t_2 + \frac{1}{C\beta} \sin\beta t_1 \cos\beta t_2$$

$$r_{21}=-\frac{1}{L\beta}\sin\beta t_1\cos\beta t_2-\frac{1}{L\beta^2\tau}\sin\beta t_1\sin\beta t_2-\frac{1}{L\beta}\cos\beta t_1\sin\beta t_2$$

$$r_{22}=\cos\beta t_1\cos\beta t_2+\frac{1}{\beta\tau}(\sin\beta t_1\cos\beta t_2-\cos\beta t_1\sin\beta t_2)-(1+\frac{2}{\beta^2\tau^2})\sin\beta t_1 \times\sin\beta t_2$$

The characteristic equation of D_1 is

$$f(\lambda)\triangleq\lambda^2-2a\{\cos\beta t_1\cos\beta t_2-(1+\frac{2}{\beta^2\tau^2})\sin\beta t_1\sin\beta t_2\}\lambda+a^2=0, \quad (6\text{-}6.32)$$

and its descriminant (denoted by D^D) is

$$D^D/4 =a\{\cos\beta t_1\cos\beta t_2-(1+\frac{2}{\beta^2\tau^2})\sin\beta t_1\sin\beta t_2\}^2-a^2. \quad (6\text{-}6.33)$$

Let λ^D be an eigen value of D_1. If $D^D<0$, the absolute value of λ^D can be written as

$$|\lambda^D|=a =e^{(t_1-t_2)/\tau}, \quad (6\text{-}6.34)$$

then the network is stable for $t_1<t_2$.

Consider the case of $D^D>0$. Since $f(0)>0$, the conditions for $|\lambda^D|<1$ are

i) $f(1)>0$ and $0<a\{\cos\beta t_1\cos\beta t_2-(1+\frac{2}{\beta^2\tau^2})\sin\beta t_1\sin\beta t_2\}<1$

or

ii) $f(-1)>0$ and $-1<a\{\cos\beta t_1\cos\beta t_2-(1+\frac{2}{\beta^2\tau^2})\sin\beta t_1\sin\beta t_2\}<0$.

Note that if the condition i) or ii) holds for $t_1>t_2$, then D^D must be negative.

The condition i) always hold in $D^D\geq 0$ (See Appendix VIII).
Next consider the condition ii). The shaded portions in Fig.6-6.4 show the domain of $D^D>0$ and $t_1<t_2$. For the region denoted by A,

$$\cos\beta t_1\cos\beta t_2-(1+\frac{2}{\beta^2\tau^2})\sin\beta t_1\sin\beta t_2<-1. \quad (6\text{-}6.35)$$

For the region denoted by B,

$$\cos\beta t_1\cos\beta t_2-(1+\frac{2}{\beta^2\tau^2})\sin\beta t_1\sin\beta t_2>1. \quad (6\text{-}6.36)$$

The intersection-points of two curves corresponding to $t_1=t_2$ and

$$\cos\beta t_1\cos\beta t_2-(1+\frac{2}{\beta^2\tau^2})\sin\beta t_1 \times\sin\beta t_2=-1 \qquad (6-6.37)$$

are given by

$$t_1=t_2=\frac{1}{\beta}\sin^{-1}\sqrt{\beta^2\tau^2/(1+\beta^2\tau^2)}. \qquad (6-6.38)$$

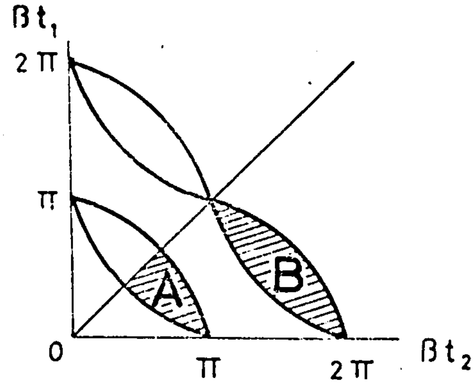


Fig.6-6.4 The region of $D^D>0$ and $t_1<t_2$.

The intersection-points of the two curves corresponding to $t_1=t_2$ and $f(-1)=0$ are given by (6-6.38). The intersection-points of the two curves corresponding to $t_1=t_2$ and

$$a\{\cos\beta t_1\cos\beta t_2-(1+\frac{2}{\beta^2\tau^2})\sin\beta t_1\sin\beta t_2\}=-1 \qquad (6-6.39)$$

are given by (6-6.38). The intersection-points of the curves corresponding to Eq.(6-6.37), (6-6.39) and $f(-1)=0$ are all on the line $t_1=t_2$. The domain of $t_1<t_2$ and

$$-1<a\{\cos\beta t_1\cos\beta t_2-(1+\frac{2}{\beta^2\tau^2})\sin\beta t_1\sin\beta t_2\}<0$$

is in the region of $t_1<t_2$ and $f(-1)>0$. Therefore the stable regions are as shown in Fig.6-6.3.

It is studied on the stability of a network, where the state equations of some circuit modes have steady state solutions, but those of others have none. It is concluded that the stability cannot be determined by the eigen values of the circuit modes only. It is neccessary to find the eigen values of the matrices D_i. No simple way to investigate the stability is found.

6-7 CONCLUSION

Some considerations on networks containing periodically operated switches is studied from a topological point of view.

In Section 6-2, it is shown that switch-operations vary the network topology. In networks satisfying some restrictions, it is easy to obtain fundamental loop and fundamental cut-set matrices without knowing the network-topology varied by switch-operations. A necessary and sufficient condition for networks with two switches to satisfy the restrictions, is obtained. A heuristic method for networks with more than two switches to satisfy the restrictions, is proposed.

In Section 6-3, the steady state solutions are obtained.

In Section 6-4, is given an explicit formula for obtaining initial values of the second kind from those of the first in RLC, RLCT and RCG networks. For active networks, the determining the initial values are very complex, but a systematic procedure for obtaining them is given. It is shown that state equations without derivatives of inputs can be obtained by choosing proper initial values. In Section 6-4.1, the initial values of the second kind are obtained on the condition that those of the fist are zero. If those of the first are nonzero, those of the second are represented by the sum of those obtained in Section 6-4.1 and 6-4.2.

In Section 6-5, it is shown that there are peculiar phenomena in the networks where initial values of the first kind are not equal to those of the second. It is shown that a complex network, whose solution is difficult to obtain, is approximated by a less complex network, in which the initial values of the first kind are not equal to those of the second.

In Section 6-6, it is shown that linear passive networks containing periodically operated switches have always steady state solutions. The stability of active networks is also discussed.

APPENDIX I[27] MATRIX PENCIL

Definition A-1.1

A matrix pencil $A+\lambda B$ is called regular if

1) A and B are square matrices of the same order n,

and

2) The determinant $|A+\lambda B|$ does not vanish identically.

In all other cases, the pencil is called singular.

Definition A-1.2

Two pencils of rectangular matrices $A+\lambda B$ and $A_1+\lambda B_1$ of the same dimension $m\times n$ connected by the equation,

$P(A+\lambda B)Q=A_1+\lambda B_1$,

where P and Q are constant square nonsingular matrices of order m and n, respectively, are called strictly equivalent.

Theorem A-1.1

A singular pencil $A+\lambda B$ of dimension $m\times n$ is strictly equivalent to

$$\begin{vmatrix} L_\varepsilon, & 0 \\ 0 & ,\tilde{A}+\lambda\tilde{B} \end{vmatrix}, \tag{A-1.1}$$

where

$$L_\varepsilon = \overbrace{\begin{vmatrix} \lambda, 1 & & & \\ & \lambda, 1 & & 0 \\ & & \ddots & \\ 0 & & & \ddots \\ & & & \lambda, 1 \end{vmatrix}}^{\varepsilon+1} \Big\} \varepsilon$$

if

$$M_\varepsilon = \left|\begin{matrix} A & & & \\ B, A & 0 & & \\ & B & \cdot & \\ 0 & & \cdot\ \cdot & A \\ & & & B \end{matrix}\right| \varepsilon$$
$$\varepsilon+1$$

is of rank $\rho_\varepsilon < (\varepsilon+1)n$, and

$$M_\alpha = \left|\begin{matrix} A & & 0 & \\ B & \cdot & & \\ & \cdot & \cdot & \\ 0 & & \cdot & A \\ & & & B \end{matrix}\right| \alpha$$
$$\alpha+1$$

is of rank $\rho_\alpha = (\alpha+1)n$ for $0 \leqq \alpha < \varepsilon$.

Theorem A-1.2

A pencil is strictly equivalent to

$$\{h \overset{g}{0}, L_{\varepsilon_{g+1}}, \cdots\cdots, L_{\varepsilon_p}, L'_{\eta_{h+1}}, \cdots\cdots, L'_{\eta_q}, A_0 + \lambda B_0\},$$

where

$$L_\varepsilon = \left|\begin{matrix} \lambda, 1 & & & \\ & & & 0 \\ & \lambda, 1 & & \\ & & \cdot\ \cdot & \\ 0 & & \cdot\ \cdot & \\ & & & \lambda, 1 \end{matrix}\right|, \quad L'_\eta = \left|\begin{matrix} \lambda & & & \\ & & 0 & \\ 1, \lambda & & & \\ & \cdot\ \cdot & & \\ 0 & \cdot & \cdot & \\ & & & 1, \lambda \end{matrix}\right| \quad \text{and}$$

$A_0 + \lambda B_0$ is a regular pencil.

Theorem A-1.3

A regular pencil $A+\lambda B$ can be reduced to a (strictly equivalent) canonical quasi-diagonal form

$$\{N^{(u_1)}, \cdots\cdots, N^{(u_n)}, J+\lambda 1\},$$

where

$$N^{(u_i)} = \begin{array}{cc} \left|\begin{array}{cccc} 1, \lambda & & 0 & \\ & 1, \lambda & & \\ & & \ddots & \\ 0 & & \ddots & \lambda \\ & & & 1 \end{array}\right| & u_i \\ u_i & \end{array}$$

and J is a Jordan matrix.

APPENDIX II APPLICATION OF PENCIL

The results in Appendix I are applied to a system of m linear differential equations of the first order in n unknown functions with constant coefficients:

$$\mathrm{A}x+\mathrm{B}\frac{d}{dt}x=f(t), \tag{A-2.1}$$

where A and B are $m\times n$ matrices, and x and $f(t)$ are vectors of dimension n.

We introduce a new unknown vector z that is connected with the old x by a linear nonsingular transformation;

$$x=Qz, \tag{A-2.2}$$

BY substituting Qz for x in Eq.(A-2.1) and multiplying (A-2.1) on the left by P, we obtain

$$\tilde{\mathrm{A}}z+\tilde{\mathrm{B}}\frac{d}{dt}z=\tilde{f}(t), \tag{A-2.3}$$

where $\tilde{\mathrm{A}}=\mathrm{PAQ}$ $\quad\tilde{\mathrm{B}}=\mathrm{PBQ}$ $\quad\tilde{f}(t)=\mathrm{P}f(t)$.

We choose the matrices P and Q such that the pencil $\mathrm{A}+\lambda\mathrm{B}$ has a canonical quasi-diagonal form;

$$\tilde{\mathrm{A}}+\lambda\tilde{\mathrm{B}}=\{h\overset{g}{0},L_{\varepsilon_{g+1}},\cdots,L_{\varepsilon_{p}},L'_{\eta_{h+1}},\cdots,L'_{\eta_{q}},N^{(u_1)},\cdots,N^{(u_s)},J+\lambda 1\} \tag{A-2.4}$$

In accordance with the diagonal blocks in (A-2.4), the system of differential equations split into $\nu=p-g+q-h+s+2$ separte systems of the form

$$0\overset{1}{z}=\overset{1}{\tilde{f}} \tag{A-2.5}$$

$$L_{\varepsilon_{g+i}}\frac{d}{dt}\overset{1+i}{z} = \overset{1+i}{\tilde{f}} \tag{A-2.6}$$

$$L'_{\eta_{h+j}}\frac{d}{dt}\overset{p-q+1+j}{z} = \overset{p-q+1+j}{f} \tag{A-2.7}$$

$$N^{(u_k)}\frac{d}{dt}\overset{p-g+q-h+1+k}{z} = \overset{p-g+q-h+1+k}{\widetilde{f}} \tag{A-2.8}$$

$$(J+\frac{d}{dt})\overset{\nu}{z}=\overset{\nu}{\widetilde{f}} \tag{A-2.9}$$

where

$$z=\begin{vmatrix} \overset{1}{z} \\ \cdot \\ \cdot \\ \overset{\nu}{z} \end{vmatrix} \qquad \widetilde{f}=\begin{vmatrix} \overset{1}{\widetilde{f}} \\ \cdot \\ \cdot \\ \overset{\nu}{\widetilde{f}} \end{vmatrix}.$$

1) The system (A-2.5) is not inconsistent if and only if

$$\overset{1}{\widetilde{f}}\equiv 0. \tag{A-2.10}$$

2) The system (A-2.6) is of the form

$$L_\varepsilon\frac{d}{dt}\ z=\widetilde{f} \tag{A-2.11}$$

or more explicitly

$$\frac{d}{dt}z_1+z_2=\widetilde{f}_1(t),\cdots\cdots\cdots,\frac{d}{dt}z_\varepsilon+z_{\varepsilon+1}=\widetilde{f}_\varepsilon(t). \tag{A-2.12}$$

Such a system is always consistent.

3) The system (A-2.7) is of the form

$$L'_\eta\frac{d}{dt}\ z=\widetilde{f} \tag{A-2.13}$$

or more explicitly

$$\frac{d}{dt}z_1=\widetilde{f}_1(t),\quad \frac{d}{dt}z_2+z_1=\widetilde{f}_2(t),\cdots\cdots,\quad \frac{d}{dt}z_\eta+z_{\eta+1}=\widetilde{f}_\eta(t),\quad z_\eta=\widetilde{f}_{\eta+1}(t). \tag{A-2.14}$$

From all the equations (A-2.14) except the first, we determine

$$\begin{aligned} z_\eta&=\widetilde{f}_{\eta+1}\\ z_{\eta-1}&=\widetilde{f}_\eta-\frac{d}{dt}\widetilde{f}_{\eta+1}\\ &\cdot\\ &\cdot\\ z_1&=\widetilde{f}_2-\frac{d}{dt}\widetilde{f}_3+\cdots\cdots+(-1)^{\eta-1}\ \frac{d^{\eta-1}}{dt^{\eta-1}}\widetilde{f}_{\eta+1}. \end{aligned} \tag{A-2.15}$$

By substituting this expression for z_1 into the first equation, we obtain the condition for consistency

$$\tilde{f}_1-\frac{d}{dt}\tilde{f}_2+\cdots\cdots+(-1)^{\eta}\frac{d^{\eta}}{dt^{\eta}}\tilde{f}_{\eta+1}=0. \tag{A-2.16}$$

4) The system (A-2.8) is of the form

$$N^{(u)}\frac{d}{dt}z=\tilde{f} \tag{A-2.17}$$

or more explicitly

$$\frac{d}{dt}z_2+z_1=\tilde{f}_1,\cdots\cdots,\frac{d}{dt}z_u+z_{u-1}=\tilde{f}_{u-1},\quad z_u=\tilde{f}_u. \tag{A-2.18}$$

Therefore we determine successively the unique solution

$$\begin{aligned} &z_u=\tilde{f}_u\\ &z_{u-1}=\tilde{f}_{u-1}-\frac{d}{dt}\tilde{f}_u\\ &\quad\vdots\\ &z_1=\tilde{f}_1-\frac{d}{dt}\tilde{f}_2+\cdots\cdots+(-1)^{u-1}\ \frac{d^{u-1}}{dt^{u-1}}\tilde{f}_u. \end{aligned} \tag{A-2.19}$$

5) The system (A-2.9) is of the form

$$Jz+\frac{d}{dt}z=\tilde{f}. \tag{A-2.20}$$

Therefore we obtain the general solution

$$z=e^{-Jt}z_0+\int_0^t e^{-J(t-\tau)}f(\tau)d\tau, \tag{A-2.21}$$

where z_0 is a vector with arbitrary values.

APPENDIX III LEMMA ON MATRIX

Lemma A-3.1

Let A and B be $m\times n$ and $n\times m$ matrices, respectively, and the following equality holds.

$$det|1+AB|=det|1+BA|. \tag{A-3.1.1}$$

Proof: Define a matrix D as

$$D=\begin{vmatrix}1,-B\\ A,\ 1\end{vmatrix}. \tag{A-3.1.2}$$

Then the matrix can be written as

$$D=\begin{vmatrix}1,0\\ A,1\end{vmatrix}\begin{vmatrix}1,\ -B\\ 0,1+AB\end{vmatrix} \quad \text{and} \quad D=\begin{vmatrix}1,-B\\ 0,\ 1\end{vmatrix}\begin{vmatrix}1+BA,0\\ A\ ,1\end{vmatrix}. \tag{A-3.1.3}$$

Then we obtain

$$det D=det|1+AB|=det|1+BA|. \tag{A-3.1.4}$$

Q.E.D.

Lemma A-3.2

Consider an equality as

$$y=Ax, \tag{A-3.2.1}$$

where A is a matrix and y and x are vectors.

Let us paritition Eq.(A-3.2.1) as

$$\begin{vmatrix}y_1\\ y\\ y_2\end{vmatrix}=\begin{vmatrix}A_{11},a_1,A_{12}\\ a_2\ ,\ a,a_3\\ A_{21},a_4,A_{22}\end{vmatrix}\begin{vmatrix}x_1\\ x\\ x_2\end{vmatrix}. \tag{A-3.2.2}$$

Then the following equality holds.

$$\begin{vmatrix}y_1\\ x\\ y_2\end{vmatrix}=\begin{vmatrix}A_{11}-\frac{1}{a}a_1a_2, & -\frac{1}{a}a_1, & A_{12}-\frac{1}{a}a_1a_3\\ -\frac{1}{a}a_2 & , \frac{1}{a} & , \quad -\frac{1}{a}a_3\\ A_{21}-\frac{1}{a}a_2a_4, & -\frac{1}{a}a_4, & A_{22}-\frac{1}{a}a_3a_4\end{vmatrix}\begin{vmatrix}x_1\\ y\\ x_2\end{vmatrix}. \tag{A-3.2.3}$$

Proof: From Eq.(A-3.2.2), we obtain

$$x = \frac{1}{a}(-a_2 x_1 + y - a_3 x_2). \tag{A-3.2.4}$$

Then we obtain

$$\begin{vmatrix} x_1 \\ x \\ x_2 \end{vmatrix} = \begin{vmatrix} 1 & 0, & 0 \\ -a_2/a, & 1/a, & -a_2/a \\ 0 & , 0, & 1 \end{vmatrix} \begin{vmatrix} x_1 \\ y \\ x_2 \end{vmatrix}, \tag{A-3.2.5}$$

and

$$\begin{vmatrix} y_1 \\ y \\ y_2 \end{vmatrix} = \begin{vmatrix} A_{11}-a_1a_2/a, & -a_1/a, & A_{12}-a_3a_4/a \\ 0 & , 1, & 0 \\ A_{21}-a_2a_4/a, & -a_4/a, & A_{22}-a_3a_4/a \end{vmatrix} \begin{vmatrix} x_1 \\ y \\ x_2 \end{vmatrix}. \tag{A-3.2.6}$$

From Eq.(A-3.2.4) and (A-3.2.6), we have Eq.(A-3.2.3).

Q.E.D.

Lemma A-3.3

Let A and B be $m\times n$ and $n\times m$ matrices, respectively. Then the following equality holds.

$$(1+AB)^{-1} = 1 - A(1+BA)^{-1}B \tag{A-3.3.1}$$

Proof: From Eq.(A-3.1.3), if the matrix D is nonsingular, we obtain

$$D^{-1} = \begin{vmatrix} 1-B(1+AB)^{-1}A, & B(1+AB)^{-1} \\ -(1+AB)^{-1}A, & (1+AB)^{-1} \end{vmatrix}, \tag{A-3.3.2}$$

and

$$D^{-1} = \begin{vmatrix} (1+BA)^{-1}, & B(1+BA)^{-1} \\ -A(1+BA)^{-1}, & 1-A(1+AB)^{-1}B \end{vmatrix}. \tag{A-3.3.3}$$

From Eq.(A-3.3.2) and (A-3.3.3), we have Eq.(3.3.1).

Q.E.D.

Lemma A-3.4

Let A and B be $n \times n$ matrices. Then the eigen values of the matrix AB are equal to those of BA.

Lemma A-3.5

A real symmetric matrix A is positive definite if and only if there exists a real nonsingular matrix P such that $A=PP'$.

Lemma A-3.6

A real symmetric matrix A written as PP' is positive semi-definite and if A is nonsingular, it is positive definite.

APPENDIX IV PROOFS OF PROPERTY 3-4.2 AND 3-4.3

Definition

The graphs obtained by the following procedure I"II " are called B-S-graphs"S-B-graphs" of G and M.

Procedure I

i) Set $G_1=G$, $M_0=M$, $i=1$.

ii) If there exists a transformer-bridge in G_i, go to Step iii). Otherwise go to Step vi).

iii) Set $M_i=M_{i-1}$[all the transformer-bridges in G_i].

iv) If there exists a transformer-self-loop in M_i, go to Step v). Otherwise Set $i=i+1$, $G_i=G_{i-1}$ and go to Step vi).

v) Set $G_{i+1}=G_i${all the trasformer-self-loops in M_i}, $i=i+1$ and go to Step ii).

vi) Stop.

The graphs G_i and M_{i-1} are B-S-graphs.

Procedure II is obtained from Procedure I by replacing bridges, self-loops, [ ] and { } with self-loops, bridges, { } and [ ], respectively.

Proof of property 3-4.2

It is assumed that $F'_{SC}C_1e_S \neq 0$. From Eq.(3-4.20), we obtain

$$\widetilde{F}'_{SC}C_1e_S=(F'_{SC}-F'_{1C}\widetilde{M}'F'_{St})C_1(e_S-F_{St}\widetilde{M}e_1)$$
$$=F'_{SC}C_1e_S-F'_{1C}\widetilde{M}'F'_{St}C_1e_S-F'_{SC}C_1F_{St}\widetilde{M}'e_1+F'_{1C}\widetilde{M}'F'_{St}C_1F_{St}\widetilde{M}e_1 \quad (A\text{-}4.1)$$

When the first term of Eq.(A-4.1) is nonzero, there exists at least one loop which consists of capacitor- and independent voltage source-branches only. Then the property holds.

Consider the case where the second term of Eq.(A-4.1) is nonzero. Then the following procedure is examined.

i) In G, choose fundamental loops which contain the capacitor-branch corresponding to the nonzero element of the term. Select the link-transformer-branches in the loops.

ii) In M, choose twig-transformer-branches in *loop(*the link-transformer-branches*)*.

iii) In G, choose link-capacitor- and link-transformer-branches in *cut-set(*the twig-transformer-branches).

iv) If link-capacitors are chosen and the corresponding elements of e_S stop. Otherwise go to Step ii).

The above procedure converges since the second term of Eq.(A-4.1) is nonzero.

Since $F'_{lG}M'F'_{St}\equiv 0$, in G there exist neither twig-resistor- nor twig-inductor

Since $F_{St}MF_{lG}\equiv 0$ and $F_{St}MF_{l\Gamma}\equiv 0$, in G ther exist neither twig-resistor- nor twig-inductor-branches in *loop(*the link-transformer-branches considered in Step i) and ii)*)*. Then the loops considered in Step i) and iii) contain capacitor- and transformer-branches only. Since the loops considered in Step ii) contain transformer-branches only, the ring sum of the loops considered in Step i), ii) and iii) contains capacitor- and twig-transformer-branches. However, the fundamental cut-sets *cut-set(*the twig-transformer-branches in the ring sum*)* in G do not contain link-capacitor-branch. Then the twig-transformer-branches are bridges in the B-S-graphs. Therefore when the second term of Eq.(A-4.1) is nonzero, the property holds.

Similarly when the third and the fourth terms are nonzero, it can be proved that this property holds.

When $F_{L\Gamma}L_2 j_\Gamma \neq 0$, it can be proved in the similar (dual) manner that this property holds.

Q.E.D.

Proof of Property 3-4.3

If the rank of F_{RC}"F_{LG}" is less than the number of the twig-capacitors"the link-inductors" for the maximum proper tree, then the matrix-product $F'_{RC}R^{-1}F_{RC}$"$F_{LG}g^{-1}F'_{LG}$" is singular. Then the matrix A is singular.

When the rank of F_{RC}"F_{LG}" is less than the number of the twig-capacitors"the link-inductors" for the maximum proper tree, ther exists at least one twig-capacitor-branch"link-inductor-branch" for a proper tree T_{pL} such that $|\bar{T}_{pL}(C)|+|T_{pL}(L)|$ is maximum. This corresponds to that there exists at least one twig-inductor-branch"link-capacitor-branch" in the maximum proper tree. Therefore we can prove this property in the same manner of the proof of Property 3-3.2.

Q.E.D.

APPENDIX V PROOFS OF PROPERTY 4-4.2 AND 4-4.3

Proof of Property 4-4.2

If the state equation has derivatives of input functions, from Eq.(4-4.21), we obtain

$A'_{SC}C_1 e_{C1} \neq 0$,

$A'_{SC}C_1 F_{C\Sigma} G_0^{-1} F'_{\Sigma\Sigma} R_{\Sigma 1}^{-1} e_{\Sigma 1} \neq 0$, or

$A'_{SC}C_1 F_{C\Sigma} G_0^{-1} j_{\Sigma 2} \neq 0$.

Let C denote a twig-capacitor corresponding to the nonzero term of the above.

1) Case of $A'_{SC}C_1 e_{C1} \neq 0$

We have

$A'_{SC} = F'_{CC} - F'_{\Sigma C} R_0^{-1} F_{\Sigma\Sigma} G_{\Sigma 2}^{-1} F'_{C\Sigma}$.

When $F'_{CC}e_{C1} \neq 0$, there exists at least one loop which consists of capacitor- and independent voltage source-branches only.

Let us consider the case where $F'_{\Sigma C} R_0^{-1} F_{\Sigma\Sigma} G_{\Sigma 2}^{-1} F'_{C\Sigma} e_{C1} \neq 0$. Then there exists a link-gyrator-branch (denoted by g_1) in *cut-set(C)*. There exists a twig-gyrator-branch (denoted by g_2) in *loop*(g_1^+). There exists a link-gyrator-branch (denoted by g_3) in *cut-set*(g_2^+) By repeating the above, for a cetain integer n, there exists a twig-gyrator-branch g_{2n} in *loop*(g_{2n-1}^+) and there exists a link-capacitor-branch S in *cut-set*(g_{2n}^+). Let us call that S is in *CL(C)* or that C is in *LC(S)*.

If there existed resistor-branches in *cut-set*(g_{2k}) or *CL*(g_{2k}) "*loop*(g_{2k-1}) or *LC*(g_{2k-1})", there would exist a proper tree which contains more capacitor-branches than the maximum proper tree. Then the property i) holds.

2) Case of $A'_{SC}C_1 F_{C\Sigma} G_0^{-1} F'_{\Sigma\Sigma} R_{\Sigma 1}^{-1} e_{\Sigma 1} \neq 0$

There exists a link-capacitor-branch in *CL(C)*. There exists a gyrator-branch in *LC(S)*. There exists a voltage source-branch in *loop(*the pair branch of the gyrator*)*. Then in the similar manner to 1), we can prove that this property i) holds.

3) Case of $A'_{SC}C_1F_{C\Sigma}G_0^{-1}j_{\Sigma 2}\neq 0$

There exists a link-capacitor-branch in *CL(C)*. There exists a twig-gyrator-branch in *LC(C)*. There exists a current source-branch in *cut-set(*the pair branch of the twig-gyrator*)*. Then this property ii) holds.

Q.E.D.

Proof of Property 4-4.3

If the rank of the matrix $F_{\Sigma C}$ is less than the number of the twig-capacitor-branches for the maximum proper tree, then the matrix $F'_{\Sigma C}R_0^{-1}F_{\Sigma C}$ is singular and the matrix A is singular.

When the rank of $F_{\Sigma C}$ is less than the number of the twig-capacitor-branches for the maximum proper tree, a proper tree which has minimum $|T_p(C)|$ contains at least one capacitor-branch. Then if there exists no proper tree in *G*{all the capacitor-branches}, the matrix A is singular.

Q.E.D.

APPENDIX VI ALGORITHM FOR A P-TREE

I) Node-labeling I

1-1 Label all the nodes v_0 adjacent to the r-node i by i in G{all r-branches}.

1-2 Label all the nodes v_1 adjacent to v_0 by i in G{all r-branches}.

1-3 Repeat Step 2 until no more labels can be assigned.

Do Node-labeling I for all the r-branches.

II) Branch-labeling I

2-1 Lable a branch by the union of the labels on the two nodes touching the branch.

III) Simplifying graph

3-1 Divide the graph into nonseparable components. (If a componoent contains less than three r-branches, Theorem 6-2.1 and 6-2.2 can be applied.)

3-2 Delete branches whose labels correpond to only one r-branch.

3-3 Transpose paralell branches which contain no r-branch into a branch.

3-4 Transpose series branches which contain no r-branches into a branch.

3-5 Transpose $d\text{-}p(i,j)$ into a branch touching the nodes i and j.

3-6 Transpose series three branches, whose center-branch is an r-branch, and where others are not r-branches, into series two branches, one of which is the r-branch and the other of which is not an r-branch.

This routine does not affect the existence of p-trees.

This routine is applied to simplify graphs after deleting and/or contracting branches in the following routines.

IV) Check d-path

Even if the number of r-branches is more than two, there is no p-tree when there exists a d-path described in Theorem 6-2.1 and 6-2.2.

1) In the case where the r-node i is identical to the r-node j, if there exists $d\text{-}p(i',j')$, there is no p-tree and stop.

2) If there exist $d\text{-}p(i,j)$, $d\text{-}p(i',j)$, $d\text{-}p(i,j')$ and $d\text{-}p(i',j')$, there is no p-tree and stop.

V) e-tree and m-tree

Definition

An e-tree is a tree which contains no r-branch (denoted by T_e).

Theorem

If there is no e-tree, there is no p-tree.

Theorem

If there is no branch, which is not an r-branch, in *cut-set(* a twig *t)* for T_e, t is contained of every p-tree if there is p-tree.

Definition

A node which is not an r-node is called a v-node.

Procedure

5-1 $ST_e=\{\phi\}$.

5-2 Choose a most labeled node v, which is not in ST_e.

5-3 Choose a minimum set of branches, such as the union of the set and ST_e is a subtree of T_e and the paths from v to the r-nodes corresponding to the label of v on the union. Set ST_e=the union.

5-4 Repeat Step 2 and 3 until there is no node as mentioned in Step 1.

5-5 If ST_e is not an e-tree, choose a set of branches such as the union of the set and ST_e is an e-tree.

Definition

An e-tree obtained in the above procedure is called an m-tree and denoted by T_m.

VI) Node-labeling II

6-1 Label a v-node adjacent to the r-node i on T_m by i.

6-2 Label a v-node adjacent to the v-node on T_m, which is labled by i.

6-3 Repeat Step 2 until no more labels can be assigned.

Do Node-labeling II for all r-nodes.

VII) Branch-labeling II

7-1 Label a branch by the union of the labels on the two nodes touching the branch.

VIII) τ-branch and f-branch

Definition

Among the twigs of T_m, a twig such that *cut-set*(a twig) contains more than one r-branch is called a τ-branch.

Theorem

If there exists no τ-branch, it does not affect the existence of the p-tree that a twig in *loop*(a link which is not an r-branch) for T_m is contracted.

Definition

Assume that t_1 and t_2 are τ-branches such that *cut-set*(t_1) contains r-branches $r_1,\cdots,r_n$ but *cut-set*(t_2) contains neither of $r_1,\cdots,r_n$. Let an r-branch in *cut-set*(t_2) be r. If there exist paths, on T_m-*loop*(r), from the node touching r to the nodes touching $r_1,\cdots,r_n$, then the branches on the paths are called f-branches for t_1.

Definition

Let the number of the r-branches in *cut-set*(t which is a τ-branch) be $N(t/T_m)$. If there is no τ-branch t_1 such as $N(t_1/T_m)>N(t/T_m)$, t is called a maximum τ-branch.

IX) Choosing p-tree

9-1 Choose an m-tree T_m.

9-2 If there exists a τ-branch, go to Step 3. Otherwise go to Step 5.

9-3 i)If there exists a τ-branch t such as *cut-set*(t) contains t and r-branches only, contract t and go to Node-labeling I. Otherwise go to Step 3-ii).

ii) If there exists no such a τ-branch, neglect l such as the branches in *loop(l)* have the same labels and do Step 3-i). Otherwise go to Step 4.

9-4 Choose a maximum τ-branch t such as the deference between the set of the r-nodes touching the r-branches in *cut-set(t)* and the set of the r-nodes corresponding to the lables of the r-branches is a maximum. Assume that *cut-set(t)* contains $r_i, r_j, \cdots$. If *cut-set(t)* contains *d-p(i,j)*, go to Step 4-i). Otherwise go to 4-ii).

i) If the union of *d-p(i,j)*, the f-braches for t and r-brances contains no loop, contract t and go to Node-labeling I. Otherwise go to Step 4-ii).

ii) Contract t and go to Node-labeling I.

9-5 Contract a twig in *loop(*a link which is not an r-branch*)* and go to Node-labeling I.

We obtain a p-tree by doing Node-labeling I, Branch-labeling I, Simplifying graph, Check d-graph, Node-labeling II, Branch-labeling II and Choosing p-tree routines, successsively.

APPENDIX VII SUBMATRICES IN Eq.(6-5.17)

$$A_{11}=-(1+F'_{r2}F_{r2})^{-1}\{(C_1^{-1}+F_{r2}C_2^{-1}F'_{r2})\boldsymbol{r}^{-1}+F_{r2}C_2^{-1}\boldsymbol{y}(1+F'_{r2}F_{r2})^{-1}$$
$$\times(F'_{r2}C_1^{-1}-C_2^{-1}F'_{r2})(C_1^{-1}+F_{r2}C_2^{-1}F'_{r2})^{-1}\}(1+F_{r2}F'_{r2})$$

$$A_{12}=-(1+F_{r2}F'_{r2})^{-1}F_{r2}C_2^{-1}\boldsymbol{y}$$

$$A_{13}=(1+F_{r2}F'_{r2})^{-1}F_{r2}C_2^{-1}H$$

$$A_{14}=-(1+F_{r2}F'_{r2})^{-1}F_{r2}C_2^{-1}H\{(1+F_{1g}F'_{1g})^{-1}(L_1^{-1}F_{1g}-F_{1g}L_2^{-1})$$
$$\times(L_2^{-1}+F'_{1g}L_1^{-1}F_{1g})^{-1}(1+F'_{1g}F_{1g})+F_{1g}\}$$

$$A_{21}=(1+F'_{r2}F_{r2})^{-1}\{(F'_{r2}C_1^{-1}-C_2^{-1}F'_{r2})(C_1^{-1}+F_{r2}C_2^{-1}F'_{r2})^{-1}F'_{r2}+1\}C^{-1}\boldsymbol{y}$$
$$\times(1+F'_{r2}F_{r2})^{-1}(F'_{r2}C_1^{-1}-C_2^{-1}F'_{r2})(C_1^{-1}+F_{r2}C_2^{-1}F'_{r2})^{-1}(1+F_{r2}F'_{r2})$$

$$A_{22}=-(1+F'_{r2}F_{r2})^{-1}\{(F'_{r2}C_1^{-1}-C_2^{-1}F'_{r2})(C_1^{-1}+F_{r2}C_2^{-1}F'_{r2})^{-1}F_{r2}+1\}$$
$$\times C_2^{-1}\boldsymbol{y}$$

$$A_{23}=(1+F'_{r2}F_{r2})^{-1}\{(F'_{r2}C_1^{-1}-C_2^{-1}F'_{r2})(C_1^{-1}+F_{r2}C_2^{-1}F'_{r2})^{-1}F_{r2}+1\}$$
$$\times C_2^{-1}H$$

$$A_{24}=-(1+F_{r2}F'_{r2})^{-1}\{(F'_{r2}C_1^{-1}-C_2^{-1}F'_{r2})(C_1^{-1}+F_{r2}C_2^{-1}F'_{r2})^{-1}F_{r2}+1\}$$
$$\times C_2^{-1}H\{(1+F_{1g}F'_{1g})^{-1}(L_1^{-1}F_{1g}-F_{1g}L_2^{-1})(L_2^{-1}+F'_{1g}L_1^{-1}F_{1g})^{-1}$$
$$\times(1+F_{1g}F'_{1g})+F_{1g}\}$$

$$A_{31}=-(1+F_{1g}F'_{1g})^{-1}\{(L_1^{-1}F_{1g}-F_{1g}L_2^{-1})(L_2^{-1}+F'_{1g}L_1^{-1}F_{1g})^{-1}F'_{1g}-1\}$$
$$\times L_1^{-1}H\{F'_{r2}-(1+F'_{r2}F_{r2})^{-1}(F'_{r2}C_1^{-1}-C_2^{-1}F'_{r2})(C_1^{-1}+F_{r2}C_2^{-1}F'_{r2})^{-1}$$
$$\times(1+F_{r2}F'_{r2})\}$$

$$A_{32}=(1+F_{1g}F'_{1g})^{-1}\{(L_1^{-1}F_{1g}-F_{1g}L_2^{-1})(L_2^{-1}+F'_{1g}L_1^{-1}F_{1g})^{-1}F'_{1g}-1\}$$
$$\times L_1^{-1}H'$$

$$A_{33}=(1+F_{1g}F'_{1g})^{-1}\{(L_1^{-1}F_{1g}-F_{1g}L_2^{-1})(L_2^{-1}+F'_{1g}L_1^{-1}F_{1g})^{-1}F'_{1g}-1\}$$
$$\times L_1^{-1}\boldsymbol{z}$$

$$A_{34}=-(1+F_{1g}F'_{1g})^{-1}\{(L_1^{-1}F_{1g}-F_{1g}L_2^{-1})(L_2^{-1}+F'_{1g}L_1^{-1}F_{1g})^{-1}F'_{1g}-1\}$$
$$\times L_1^{-1}\boldsymbol{z}(1+F_{1g}F'_{1g})^{-1}(L_1^{-1}F_{1g}-F_{1g}L_2^{-1})(L_2^{-1}+F'_{1g}L_1^{-1}F_{1g})^{-1}(1+F'_{1g}F_{1g})$$

$$A_{41}=(1+F'_{1g}F_{1g})^{-1}F'_{1g}L_2^{-1}H'\{F'_{r2}-(1+F'_{r2}F_{r2})^{-1}(F'_{r2}C_1^{-1}-C_2^{-1}F'_{r2})$$
$$\times(C_1^{-1}+F_{r2}C_2^{-1}F'_{r2})^{-1}(1+F_{r2}F'_{r2})\}$$

$$A_{42}=(1+F'_{1g}F_{1g})^{-1}F'_{1g}L_1^{-1}H'$$

$$A_{43}=(1+F'_{1g}F_{1g})^{-1}F'_{1g}L_1^{-1}\boldsymbol{z}$$

$$A_{44}=-(1+F'_{1g}F_{1g})^{-1}\{F'_{1g}L_1^{-1}\boldsymbol{z}(1+F_{1g}F'_{1g})^{-1}(L_1^{-1}F_{1g}-F_{1g}L_2^{-1})$$
$$\times(L_2^{-1}+F'_{1g}L_1^{-1}F_{1g})^{-1}+(L_2^{-1}+F'_{1g}L_1^{-1}F_{1g})\boldsymbol{g}^{-1}\}(1+F'_{1g}F_{1g})$$

APPENDIX VIII SUPPLEMENT OF SECTION 6-6

Lemma

If $D^D>0$ and $t_1<t_2$, the following inequalities hold.

$$f(1)>0 \tag{A-8.1}$$

$$0<a\{\cos\beta t_1\cos\beta t_2-(1+\frac{2}{\beta^2\tau^2})\sin\beta t_1\sin\beta t_2\}<1. \tag{A-8.2}$$

Proof: Since $a>0$ and $\alpha>1$, we obtain

$$a\{\cos\beta t_1\cos\beta t_2-(1+\frac{2}{\beta^2\tau^2})\sin\beta t_1\sin\beta t_2\}>0.$$

The following equality holds.

$$a\{\cos\beta t_1\cos\beta t_2-(1+\frac{2}{\beta^2\tau^2})\sin\beta t_1\sin\beta t_2\}$$

$$=e^{(t_1-t_2)/\tau}\{(1+\frac{1}{{}^{2}{}^{2}})\cos\beta(t_1+t_2)-\frac{1}{\beta^2\tau^2}\cos\beta(t_1-t_2)\}.$$

We define $g(x)$ as

$$g(x)=e^{x/\tau}\{(1+\frac{1}{\beta^2\tau^2})\cos\beta(t_1+t_2)-\frac{1}{\beta^2\tau^2}\cos\beta x\},$$

where $x=t_1-t_2$.

Let us define $g_1(x)$ as

$$g_1(x)=e^{x/\tau}\{(1+\frac{1}{\beta^2\tau^2})-\frac{1}{\beta^2\tau^2}\cos\beta x\},$$

then we obtain

$$g_1(x)>g(x) \quad \text{and} \quad g_1(0)=1.$$

Differentiating $g_1(x)$ with respect to x, we obtain

$$g_1'(x)=e^{x/\tau}\{\frac{1}{\tau}(1+\frac{1}{\beta^2\tau^2})-\frac{1}{\beta\tau^2}\sqrt{(1+\frac{1}{\beta^2\tau^2})}\cos(\beta x+\theta),$$

where $\tan\theta=\beta\tau$.

There is no value of x such that $g_1'(x)=0$. Since $g_1'(0)>0$, we obtain $g_1'(x)>0$. Then we obtain

$$g_1(x)<1 \quad \text{for} \quad x<0.$$

Then the inequality (A-8.2) holds.

Next we consider the inequality (A-8.1). We define $h(x)$ as

$$h(x)=f(1)$$
$$=1-2\{(1+\frac{1}{\beta^2\tau^2})\cos\beta(t_1+t_2)-\frac{1}{\beta^2\tau^2}\cos\beta x\}e^{x/\tau}+e^{2x/\tau},$$

where $x=t_1-t_2$.

Let us define $h_1(x)$ as

$$h_1(x)=1-2\{(1+\frac{1}{\beta^2\tau^2})-\frac{1}{\beta^2\tau^2}\cos\beta x\}e^{x/\tau}+e^{2x/\tau}.$$

Then we obtain

$$h_1(x)<h(x) \quad \text{and} \quad h_1(0)>1.$$

Defferentiating $h_1(x)$ with respect to x, we obtain

$$h_1'(x)=\frac{2}{\tau}e^{x/\tau}\{e^{x/\tau}-(1+\frac{1}{\beta^2\tau^2})+\frac{1}{\beta\tau}\sqrt{(1+\frac{1}{\beta^2\tau^2})}\cos(\beta x+\theta)\},$$

where $\tan\theta=\beta\tau$.

We obtain x_1 such that $e^{x/\tau}+\frac{1}{\beta\tau}\sqrt{(1+\frac{1}{\beta^2\tau^2})}\cos(\beta x+\theta)$ is maximum as

$$e^{x_1/\tau}=(1+\frac{1}{\beta^2\tau^2})\sin(\beta x_1+\theta).$$

Then we obtain $h_1'(x_1)<0$. Therefore we obtain

$$h_1(x)>1 \quad \text{for} \quad x<0.$$

Consequently the inequality (A-8.1) holds.

Q.E.D.

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